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The spectrum of BPS branes on a noncompact Calabi-Yau

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Abstract: We begin the study of the spectrum of BPS branes and its variation on lines of marginal stability on $\mathcal{O}_{\mathbb{P}^2}(-3)$, a Calabi-Yau ALE space asymptotic to $\mathbb{C}^3/\mathbb{Z}_3$. We show how to get the complete spectrum near the large volume limit and near the orbifold point, and find a striking similarity between the descriptions of holomorphic bundles and BPS branes in these two limits. We use these results to develop a general picture of the spectrum. We also suggest a generalization of some of the ideas to the quintic Calabi-Yau.

Keywords: D-branes, Superstring Vacua.

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Contents

1. Introduction

D-branes in Calabi-Yau backgrounds of string theory have been studied in a number of recent works, both to obtain the spectrum of BPS states in these backgrounds and because their world-volume theories are $d=4$ $\mathcal{N}=1$ gauge theories which could appear in realistic models and which naturally encode the geometry and especially the moduli spaces of vector bundles on a CY. Background and a list of references can be found in [1]; more recent works include [2 − 7].

A primary question is to classify the BPS branes for a given CY. Because the bulk theory has $\mathcal{N}=2$ supersymmetry, the spectrum of BPS branes will depend on the moduli controlling the BPS central charges (the Kähler moduli for B branes) and the classification must reflect this dependence.

In this work we begin the study of the spectrum of BPS branes on the three complex dimensional ALE space $\mathcal{O}_{\mathbb{P}^2}(-3)$. The stringy geometry of this space can be determined using local mirror symmetry [8, 9] and is quite similar to compact CY's such as the quintic; the Kähler moduli space has a conifold point and a point of enhanced discrete symmetry where the $\mathbb{P}^2$ degenerates to a $\mathbb{C}^3/\mathbb{Z}_3$ orbifold singularity. Using these results, one can relate D-brane charges at different points in moduli space [10], as we review in section 2.

This example has many advantages which will allow us to get a fairly complete picture. First, the spectrum of BPS branes in the large volume limit follows from known mathematical results, namely the classification of stable vector bundles on $\mathbb{P}^2$ [11]. This classification proceeds in two steps: one first constructs all holomorphic bundles on $\mathbb{P}^2$, and then identifies a subset as stable. We review the general construction of holomorphic bundles on $\mathbb{P}^2$ due to Beilinson [12], and related topics in section 3.

In section 4 we review results from the general analysis of D-branes on orbifolds [13]. This analysis provides quiver gauge theories describing all BPS branes at and near the orbifold point as supersymmetric vacua. The problem of finding such vacua also proceeds in two steps: first, find F-flat configurations, then minimize the D term contributions to the potential. As it turns out, there is a very relevant branch of mathematics, the theory of quiver representations (we provide an introduction in the appendix), in which the problem again becomes to find all "holomorphic objects" and then find the stable subset. This will allow us to give a good picture of the spectrum of branes at and near the orbifold point.

It will turn out that the holomorphic objects at the two points in moduli space are essentially the same, in a sense we describe in detail. The point is that the orbifold quiver theory turns out to reproduce Beilinson's general construction of bundles on $\mathbb{P}^2$. This allows us not only to identify D-brane charges with Chern classes of bundles, but to identify the entire moduli spaces of these configurations and even explicitly construct the bundle corresponding to any point in the quiver theory moduli space.

In section 5 we compare the BPS spectrum in the large volume limit with that in the neighborhood of the orbifold point. In both cases, the BPS spectrum consists of stable holomorphic objects, but with a different definition of stability in the two regimes: μ-stability in the large volume limit, and θ-stability [14] near the orbifold point. We describe the spectrum near the orbifold point in some detail; it is very dependent on the particular

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line coming in, and on some lines (such as the one to the conifold point) is significantly smaller than the large volume spectrum but still infinite.

Both of these definitions of stability are special cases of a proposed condition determining the BPS branes at general points in moduli space [15]. This starts from the "decoupling conjecture" of [16]. For B branes, this states that the holomorphic structure (field content and F-flatness conditions) of such a world-volume theory depends only on the complex structure of the CY, while the D-flatness conditions depend only on the Kähler moduli. Further discussion and arguments for these claims can be found in [17].

Given decoupling, we can assert that the problem of finding BPS branes at any point in moduli space can be studied by the same two-step procedure: identify a category of holomorphic objects, which at least locally does not depend on Kähler moduli, and then for a specific point in Kähler moduli space identify the stable subset. (See [17] for an explanation of the relevance of the term "category" here.) In [15] a simple criterion (called Π-stability) was proposed, which reduces to the known physical stability conditions in all examples studied so far.

In section 6 we apply Π-stability to develop a picture of how the BPS spectrum evolves from the orbifold point to the large volume limit. It will turn out that there are an infinite number of lines of marginal stability; we will show how a specific line or brane decay could be studied. We will also exhibit lines of marginal stability arbitrarily close to the large volume limit. We attempt to incorporate all this information in a general picture of the spectrum.

Beilinson's construction of holomorphic bundles applies to projective space of any dimension, and the results of [16] for D-branes on the quintic bear a striking relation to the construction for $\mathbb{P}^4$, which we explain in section 7.

Section 8 contains conclusions.

Some general points of notation and convention. We are studying the problem of brane configurations in type-II string theory in the classical limit, zero string coupling but non-zero string length, described by CFT with sphere and disk world-sheets. We always refer to branes by the dimension of the cycle they wrap in the CY, but usually describe their world-volume theories in terms of the $d = 4$, $\mathcal{N} = 1$ supersymmetric gauge theory which would be obtained by letting them extend in the $3 + 1$ Minkowski dimensions.

2. The stringy moduli space of $\mathcal{O}_{\mathbb{P}^2}(-3)$

We are interested in a noncompact CY $\mathcal{M}$ which can be defined in two ways. One way is as a line bundle over the complex projective plane $\mathbb{P}^2$, whose first Chern class $\mathrm{c}_1 = -3$ is chosen to produce $\mathrm{c}_1(\mathcal{M}) = 0$. The homology is that of $\mathbb{P}^2$ with $b_0 = b_2 = b_4 = 1$; let ω be the generator of $H^2(\mathbb{Z})$. The Ricci flat metric on this space was written down explicitly by Calabi [18] and should be a good description for large Kähler class (in string units). In this metric it is clear that compact minimal volume surfaces embed entirely into the $\mathbb{P}^2$.

The same space can be defined as the blowup of the orbifold singularity $\mathbb{C}^3/\mathbb{Z}_3$ and indeed Calabi's metric is asymptotically locally euclidean (ALE) to this flat geometry. On the other hand, for small Kähler class the curvature of the $\mathbb{P}^2$ is large, so in this regime the precise form of the metric and other geometric predictions cannot be trusted a priori.

The stringy moduli space is a Riemann sphere with three punctures. One of these is the large volume limit for which a good coordinate is $z = \exp 2\pi i(B + iJ)$ where $B + iJ$ is the complexified Kähler form. In this coordinate system the large volume limit is at $z = 0$. The observables of most interest for us are the masses of BPS $2k$-branes which in this limit will be

$$m_{2k} = |\Pi_{2k}| \to \left| \frac{1}{k!}(B + iJ)^k \right| . \tag{2.1}$$

These receive world-sheet instanton corrections which can be expressed as a Taylor series in z with radius of convergence 1 determined by a singularity with logarithmic monodromy at $z = 1$, the conifold point. Further analytic continuation reaches a point ($z = \infty$) with $\mathbb{Z}_3$ monodromy, the orbifold point.

The periods Π_{2k} can be computed using local mirror symmetry [9, 8] and satisfy the following differential equation (where $\theta_z = z\frac{d}{dz}$):

$$\left[\theta_z^3 - z\left(\theta_z + \frac{1}{3}\right)\left(\theta_z + \frac{2}{3}\right)\theta_z\right]\Pi = 0 . \tag{2.2}$$

There are three linearly independent solutions (including $\Pi_0 = 1$).

Let us define a basis Π_{2k} in which the mass of a brane with charges Q_{2k} is $m = |Q \cdot \Pi|$. We define these charges in terms of the topological class of a D4-brane configuration: this is determined by the number of branes N wrapping $\mathbb{P}^2$ and the Chern character $\operatorname{tr} e^F$ of the vector bundle E it carries (which is a source of RR charge in the usual way). They are then

$$Q_4 = r(E) = N\,; \qquad Q_2 = \int_\Sigma \mathrm{c}_1(E)\,; \qquad Q_0 = \int \mathrm{ch}_2(E)\,. \tag{2.3}$$

(note that charges are not integral in this basis).

One property of this basis is that the large volume monodromy $B \to B + 1$ is

$$M_\infty^{(LV)} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \frac{1}{2} & 1 & 1 \end{pmatrix} \tag{2.4}$$

acting (on the right) on the charge vector $(Q_0\ Q_2\ Q_4)$. (it has the same effect as the shift $\int_\Sigma F \to \int_\Sigma F + 1$). This operation corresponds to tensoring the vector bundle V with a line bundle, an operation which (at large volume) preserves the attributes of the brane (stability and the dimension of the moduli space) which enter our discussion.

Asking that this monodromy acts in the corresponding way on the periods determines the basis up to a monodromy of the same form and up to an overall constant shift of Π_4 (in terms of the charges, there is an ambiguity $Q_0 \to Q_0 - cQ_4$). A basis for which this is true is defined in [10]: it is[1] 1, t and t_d defined by the large volume asymptotics $t \sim B + iJ$ and $t_d \sim \frac{1}{2}t^2 + \frac{1}{8}$.

[1]Note that what we call t is t_b in [10], also note that t can sometimes be viewed as a two form.

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This determines $\Pi_2 = t$ and $\Pi_4 = t_d + c$ for some constant c. One can determine $c = 0$ by [10, 19] matching the asymptotic formula

$$Z(E) = -\int e^{-t}\mathrm{ch}(E)\sqrt{\frac{\hat{A}(T)}{\hat{A}(N)}}\,, \tag{2.5}$$

where t is the complexified Kähler form and T and N are the tangent and normal bundles of the blown up $\mathbb{P}^2$. One can also check the mass formula — curvature terms in the D4 world-volume action [20] produce a mass shift which is 1/8 the D0 mass. These two determinations are surely related by supersymmetry.

We turn to the other singular points. In this charge basis, the monodromy around the conifold point is

$$M_c^{(LV)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.6}$$

with $\Pi_4 = t_d = 0$ and $t = -3t_d \log(1-z) + O(1)$. This is the usual monodromy associated with a massless BPS state, here the pure (trivial vector bundle) D4-brane.

The orbifold point has $\mathbb{Z}_3$ monodromy; in this basis

$$M_o^{(LV)} = M_c^{(LV)} M_\infty^{(LV)} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & -2 & -3 \\ \frac{1}{2} & 1 & 1 \end{pmatrix}. \tag{2.7}$$

Here $t_d = 1/3$ and $t = -1/2$.

Considerations at this point are simpler in terms of a basis consisting of the D4-brane and its $\mathbb{Z}_3$ images. These will turn out to be the elementary "fractional branes," as was shown in [10] and as we discuss in section 4. Their periods have the asymptotic form[2]

$$\Pi_1 = \frac{1}{2}t^2 + t + \frac{5}{8} + \cdots, \qquad \Pi_2 = -t^2 - t + \frac{1}{4} + \cdots, \qquad \Pi_3 = \frac{1}{2}t^2 + \frac{1}{8} + \cdots. \tag{2.8}$$

Comparing (2.5) and (2.8) we can express the large volume charges $(Q_4\ Q_2\ Q_0)$ in terms of the orbifold charges $(n_1\ n_2\ n_3)$

$$Q_0 = -\frac{n_1 + n_2}{2}; \qquad Q_2 = n_1 - n_2; \qquad Q_4 = -n_1 + 2n_2 - n_3. \tag{2.9}$$

The monodromy matrices in the fractional brane basis are

$$M_o^{(O)} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \tag{2.10}$$

$$M_c^{(O)} = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.11}$$

[2] We changed the numbering of the branes from [10], for reasons which will become clear.

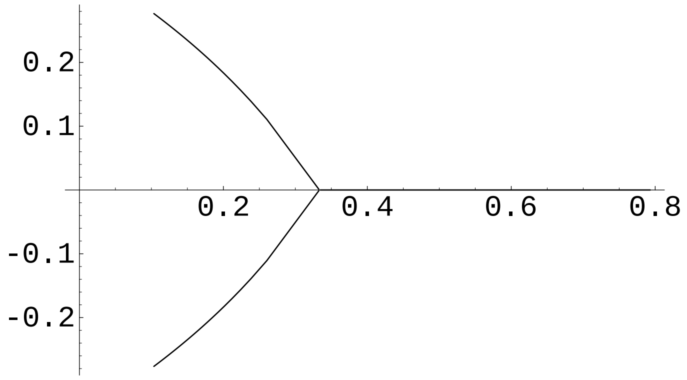

Figure 1: The evolution of the periods along the negative ξ axis.

and

$$M_{\infty}^{(O)} = \begin{pmatrix} 0 & 1 & 0 \\ -3 & 3 & 1 \\ 1 & 0 & 0 \end{pmatrix}. \tag{2.12}$$

Let L be the negative $\xi = 1/z$ axis, a line connecting the orbifold point $\xi = 0$ to the large volume limit $z = 0$ which intersects no singularities (it is opposite to the conifold point).

The evolution of the central charges of the three fractional branes along L away from the orbifold point is plotted in figure 1. All three periods are $1/3$ at the orbifold point. The periods Π_1 and Π_3 are complex conjugate, while the period Π_2 increases along the positive real axis.

Note in particular that the relative periods encircle zero, which is the basic criterion for significant changes in the BPS spectrum (and other physics) between the two limits.

Note also that any charge vector for which $\sum n_i = 0$ potentially describes a BPS brane which would become massless at the orbifold point. Many of these branes are known to exist in the large volume limit, the simplest example being $(1\ 0\ -1)$ which is the two-brane with $c_1 = 0$ (no magnetic flux). This might be a little surprising as there is a sense in which $B = 1/2$ at the orbifold point [9] (this was our earlier $t = -1/2$). However, the other world-volume couplings conspire to cancel this contribution to the mass.

Assuming the theory is consistent, because the orbifold CFT is non-singular, all of these branes will have to decay before reaching the orbifold point. Thus we can already see some significant differences between the large volume and stringy spectrum.

3. Branes in the large volume limit

In this section we describe the spectrum of BPS branes in the large volume limit. We first note that in this paper we will only discuss the classical (zero string coupling) spectrum; in other words branes which can be defined as boundary states in conformal field theory. We summarize some standard notations from algebraic geometry in appendix B.

The branes in the large volume limit fall into the following three classes:

- The D0-brane, with $Q_0 = 1$, $Q_2 = Q_4 = 0$.
- D2-branes ($Q_4 = 0$) are described by holomorphic and antiholomorphic curves in $\mathbb{P}^2$ (characterized by degree) carrying a vector bundle (characterized by c_1). The homogenous equations $x^n + y^n + z^n = 0$ in $\mathbb{P}^2$ represent all positive integral degrees and thus all integral $Q_2 \neq 0$ appear ($Q_2 < 0$ appear as antiholomorphic curves). They are curves of genus $1 + n(n-3)/2$ and can carry line bundles of arbitrary integral c_1; thus all Q_0 can appear.
$$n = Q_2\,, \qquad \mathrm{c}_1 = Q_0\,. \tag{3.1}$$
- D4-branes are described by μ-stable holomorphic vector bundles on $\mathbb{P}^2$, characterized topologically by the rank r, c_1 and c_2. These are related to Q_{2k} as $Q_4 = r$, $Q_2 = \mathrm{c}_1$ and $Q_0 = \mathrm{ch}_2$.

Although every triple $(r, \mathrm{c}_1, \mathrm{c}_2)$ with $r > 0$ appears as a topological bundle, not all of these can support stable bundles. The importance of this condition for us comes from Donaldson's theorem on the existence of solutions to the anti-self-dual Yang-Mills equation [21]. An unstable bundle will not admit a solution, and thus any such four-brane will not be BPS. A semistable bundle will only admit reducible self-dual connections, which admit more than one independent covariantly constant scalar $D_i X = 0$. These include moduli for motion in $\mathbb{R}^{3,1}$ and thus these configurations can be broken up into several branes.

In the following, instead of discussing holomorphic vector bundles, we will be a bit more general and use coherent sheaves, mostly because these behave much better in many of the mathematical constructions. Physically, this more or less corresponds to including subbranes supported on submanifolds of the original brane and is useful for describing point-like instantons and quantum bound states [22]. We also note that, although it is not a priori obvious, it turns out that for $\mathbb{P}^2$, a stable coherent sheaf at generic points in its moduli space is in fact a bundle [11].

3.1 Stable bundles on $\mathbb{P}^2$

We proceed to quote the classification of stable bundles on $\mathbb{P}^2$. The two invariants which determine stability are the *slope*
$$\mu = \frac{\mathrm{c}_1}{r} = \frac{Q_2}{Q_4} \tag{3.2}$$
and *discriminant*
$$\Delta = \frac{1}{2r^2}(2r\mathrm{c}_2 - (r-1)\mathrm{c}_1^2) = \frac{1}{2Q_4^2}(Q_2^2 - 2Q_0 Q_4)\,. \tag{3.3}$$

(We will sometimes refer to D0 and D2 branes as having infinite slope and discriminant.)z

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A sheaf E is μ-stable (respectively μ-semistable) if it is torsion-free[3] and for every coherent sub-sheaf E' of rank $0 < r' < r$, we have $\mu(E') < \mu(E)$ (respectively $\leq$). The geometric origin of this condition is explained in [21, 23, 11]. The basic idea of Donaldson's theorem is to find a solution of the Yang-Mills equations as a minimum of the action on an orbit of the complexified gauge group. Such a minimum will exist if the orbit is closed; conversely it can be shown that if the orbit is not closed, the minimum lies off of the orbit and no solution exists. Stability is the necessary and sufficient condition for the orbit to be closed.

It is interesting to note thatt he condition $\mu(E') < \mu(E)$ on $\mathbb{P}^2$ can be expressed in terms of an intersection form on $\mathbb{C}^3/\mathbb{Z}_3$: it is

$$\mathrm{sgn}(Q_4 Q_4')(Q_2' Q_4 - Q_2 Q_4') < 0\,. \tag{3.4}$$

The discriminant appears in Bogomolov's inequality

$$\text{semistability} \Rightarrow \Delta \geq 0 \tag{3.5}$$

as well as in the formula for the expected complex dimension of moduli space,

$$d = H^1(E^* \otimes E) = 1 + r^2(2\Delta - 1)\,. \tag{3.6}$$

Bogomolov's inequality can be proven by assuming an anti-self-dual Yang-Mills connection F exists and doing a straightforward computation using the inequality (for hermitian $M = iF$) $(\mathrm{tr}\, M)^2 \leq N\, \mathrm{tr}\, M^2$ [23]. It is a necessary but not sufficient condition for this connection to exist. The expected dimension can be computed using the index theorem and it is a non-trivial result of the theory that, when the connection exists, the moduli space is (locally) a manifold of the expected dimension. Thus $d \geq 0$ gives a stronger necessary condition.

In fact there are two cases:

- $d = 0$ (then $\Delta < 1/2$). This is an "exceptional bundle." It can be shown that, for a given slope α, there exists at most one exceptional bundle, and its rank is the least positive integer such that $r\alpha \in \mathbb{Z}$. Then (3.6) determines c_2 and Δ_α. Let $\mathcal{E}$ be the set of slopes of exceptional bundles; we will describe it in section 5.

- $d > 0$ (then $\Delta \geq 1/2$) are "regular bundles." The additional necessary condition for this bundle to exist is

  for all $\alpha \in \mathcal{E}$ such that $r_\alpha < r$ and $|\mu - \alpha| < 3$, we have

  $$\Delta + \Delta_\alpha \geq P(-|\mu - \alpha|)\,, \tag{3.7}$$

  where the Hilbert polynomial is given by $P(\nu) = 1 + \frac{1}{2}(\nu^2 + 3\nu)$.

[3]This more or less means that no D$p - 2$-subbranes appear.

This condition follows directly from the μ-stability condition and the Riemann-Roch formula:

$$\chi(E_1, E_2) \equiv \sum_i (-1)^i \dim_{\mathbb{C}} \mathrm{Ext}^i(E_1, E_2)$$
$$= r_1 r_2 (P(\mu_2 - \mu_1) - \Delta_1 - \Delta_2) \,. \tag{3.8}$$

Suppose $\alpha > \mu$; then a stable bundle E of slope μ must satisfy $H^0(E_\alpha, E) = 0$ (or else the image of such a morphism would be a destabilizing subbundle, see section 6). Similarly if $\alpha - \mu < 3$, then $\mathrm{Ext}^2(E_\alpha, E) = 0$, or else Serre duality tells us that $H^0(E_{-\alpha}, E^* \otimes \Omega^{-1}) \neq 0$) destabilizes a related stable bundle. This tells us that $\chi(E_\alpha, E) \leq 0$ which translates into (3.7).

We have now given a set of conditions which are clearly necessary for stability. They are also sufficient, but the proof is rather complicated and we refer to [11]. They can also be made much more concrete, but we postpone this discussion to section 5.

3.2 Translation into orbifold basis

It will be useful to have these formulas translated into the orbifold basis. The slope, discriminant and the expected dimension of the moduli space then take the form

$$\mu = \frac{n_2 - n_1}{n_1 - 2n_2 + n_3}$$
$$\Delta = \frac{-n_1 n_2 - n_1 n_3 - n_2 n_3 + 3n_2^2}{2(n_1 - 2n_2 + n_3)^2}$$
$$d = 1 - \frac{1}{2} n^t \cdot C^{LV} \cdot n \tag{3.9}$$

with the “Cartan” matrix

$$C^{LV} = \begin{pmatrix} 2 & -3 & 3 \\ -3 & 2 & -3 \\ 3 & -3 & 2 \end{pmatrix} . \tag{3.10}$$

Exceptional bundles satisfy $n^t \cdot C^{LV} \cdot n = 2$, while stable nonexceptional bundles E must satisfy $n^t \cdot C^{LV} \cdot n \leq 0$ and the second condition above: for every $\alpha \in \mathcal{E}$ with $|\alpha - \mu| < 3$, $\chi(E_\alpha, E) \leq 0$ where

$$\chi(E, E') = n \cdot \begin{pmatrix} 1 & -3 & 3 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix} \cdot n' \,. \tag{3.11}$$

3.3 Beilinson monads

Sheaves on $\mathbb{P}^2$ and their moduli spaces can be described explicitly by use of “Beilinson monads” [12, 11, 24 − 26]. This is much like the linear sigma model construction of vector bundles popular in the physics literature [27].

We consider a quiver (see appendix) with three vertices V_1, V_2 and V_3 and six arrows: X^i for $1 \leq i \leq 3$ run from V_1 to V_2, and Y^i for $1 \leq i \leq 3$ run from V_2 to V_3 We furthermore impose the relations

$$Y^i X^j = Y^j X^i \,. \tag{3.12}$$

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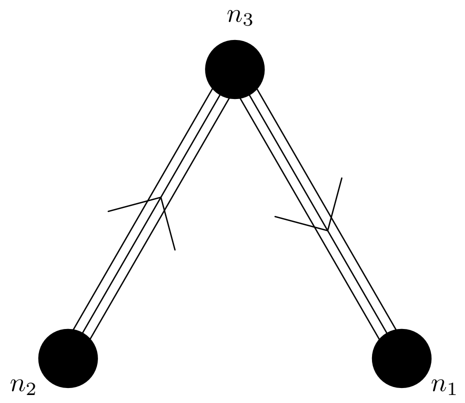

Figure 2: The Beilinson quiver.

As is easy to check, the expected dimension of the moduli space of such a quiver is

$$d = 1 - \frac{1}{2} n^t \cdot C^{LV} \cdot n \tag{3.13}$$

where $n_i = \dim V_i$, exactly the dimension of the moduli space of bundles on $\mathbb{P}^2$ expressed in the basis (2.9). This is not a coincidence — the moduli spaces of these quivers and the moduli space of bundles with these charges are the same.

In fact, there is a detailed correspondence between representations of this quiver and coherent sheaves on $\mathbb{P}^2$, which we now explain. A representation $(n_1\ n_2\ n_3)$ of the quiver can be rewritten as a complex,

$$0 \to \mathbb{C}^{n_1} \otimes \Lambda^2 V^* \xrightarrow{X} \mathbb{C}^{n_2} \otimes V^* \xrightarrow{Y} \mathbb{C}^{n_3} \to 0\,, \tag{3.14}$$

where $V = \mathbb{C}^3$ with basis (e_i). The maps are the natural contractions with $X^i e_i$ and $Y^i e_i$. The relations of the quiver are then equivalent to $Y \cdot X = 0$.

We take $\mathbb{P}^2 = \mathbb{P}(V)$, so the usual projective coordinates z^i on $\mathbb{P}^2$ are elements of V^* (linear functions on V). They are also sections of $\mathcal{O}(1)$, so we can write the exact sequence (dual to the Euler sequence [28])

$$0 \to \Omega(1) \to \mathcal{O} \otimes V^* \xrightarrow{*} \mathcal{O}(1) \to 0 \tag{3.15}$$

where $\xrightarrow{*}$ is the evaluation map (multiplication of the two factors). Translating into components, this just says that Ω, which is the cotangent bundle on $\mathbb{P}^2$, has local sections which can be written as $\psi_i(z) dz^i$ satisfying $z^i \psi_i(z) = 0$.

Thus, we can tensor (3.14) with $\mathcal{O}$ and restrict $\mathcal{O} \otimes \Lambda^i V^*$ to $\Omega^i(i)$ (and use $\Lambda^2\Omega(2) \cong \mathcal{O}(-1)$), to get a new complex

$$0 \to \mathbb{C}^{n_1} \otimes \mathcal{O}(-1) \xrightarrow{\hat{X}} \mathbb{C}^{n_2} \otimes \Omega^1(1) \xrightarrow{\hat{Y}} \mathbb{C}^{n_3} \otimes \mathcal{O} \to 0\,, \tag{3.16}$$

with maps $\hat{X}$ taking $\psi_{[ij]}$ to $X^j \psi_{[ij]}$ and $\hat{Y}$ taking ψ_i to $Y^i \psi_i$.

This complex is the Beilinson monad. If we start with a Schur representation of the Beilinson quiver, this sequence is exact at the first and third node, and the cohomology at the second node is a simple coherent sheaf E with Chern character

$$\mathrm{ch}(E) = n_2\mathrm{ch}(\Omega^1(1)) - n_1\mathrm{ch}(\Omega^2(2)) - n_3\mathrm{ch}(\mathcal{O})\,. \tag{3.17}$$

One can check that

$$\mathrm{ch}(\mathcal{O}(-1)) = 1 - \omega + \frac{1}{2}\omega^2\,, \qquad \mathrm{ch}(\Omega(1)) = 2 - \omega - \frac{1}{2}\omega^2\,, \qquad \mathrm{ch}(\mathcal{O}) = 1 \tag{3.18}$$

so that

$$\mathrm{ch}(E) = Q_4 + Q_2\omega + Q_0\omega^2 \tag{3.19}$$

where we take the relation between n_i and Q_{2p} to be exactly (2.9). This match between the result from mirror symmetry and the Beilinson construction is quite remarkable and gives hope for similar constructions.

We now discuss how to obtain a quiver from a sheaf E. The simplest case is to suppose its slope satisfies $-1 < \mu \leq 0$ (this is sometimes called a "normalized" sheaf, and can be arranged by tensoring with a line bundle). Under this condition, one can show that $h^0(E(n)) = h^2(E(n)) = 0$ for $-2 \leq n \leq 0$, and applying the Riemann-Roch theorem one finds

$$\begin{aligned} n_3 &= h^1(E) = -\chi(E) = r(E) + \frac{3}{2}\mathrm{c}_1(E) + \mathrm{ch}_2(E)\,, \\ n_2 &= h^1(E(-1)) = \mathrm{c}_1(E) + r(E) - \chi(E)\,, \\ n_1 &= h^1(E(-2)) = 2\mathrm{c}_1(E) + r(E) - \chi(E) \end{aligned} \tag{3.20}$$

which is exactly the inverse of the relation (3.19) above.

The quiver is then obtained by taking $H^1(E(-2))$, $H^1(E(-1))$ and $H^1(E)$ as representation spaces for the vertices. The maps X^i and Y^i are the restrictions to cohomology of the action of multiplication by z^i on a section of $E(n)$, giving a section of $E(n+1)$. These maps obviously satisfy the relations $Y^iX^j = Y^jX^i$.

There is then a theorem of Beilinson that states that the cohomology of the corresponding monad (3.16) is exactly the sheaf E. The proof, although short, uses a good deal of homological algebra, which we will not get into here (see [11], pp. 182–183).

This goes a long way towards establishing a one-to-one relation between quiver representations and sheaves, but the relation we just described is not completely general. It can hold only when the charges satisfy the inequality

$$r = 2n_2 - n_1 - n_3 > 0 \tag{3.21}$$

as otherwise $\ker v/\operatorname{im} u$ is degenerate. This is guaranteed if the sheaf was normalized, so this suffices to construct all moduli spaces (since these are invariant under tensoring with line bundles), but does not give a general relation.

Beilinson's theorem in fact applies to all sheaves and states that the bundle E can be recovered as the cohomology of a spectral sequence with initial term

$$E^1_{p,q} = H^q(\mathbb{P}^2, E(p)) \otimes \Omega^{-p}(-p)\,. \tag{3.22}$$

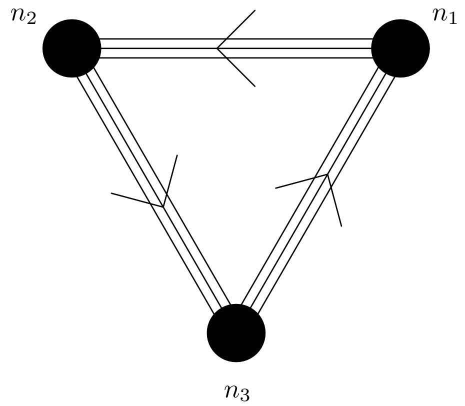

Figure 3: The $\mathbb{Z}_3$ quiver.

If $h^q \neq 0$ for a single q, this will reduce to the cohomology of a complex, as we described. This provides a partial generalization of the relation, to the cases where only $h^0 \neq 0$ (true after for tensoring with $\mathcal{O}(n)$ for some $n \gg 0$), and only $h^2 \neq 0$ (true for some $n \ll 0$), which will also be used below.

4. Branes at the orbifold point

4.1 The $\mathbb{C}^3/\mathbb{Z}_3$ quiver gauge theory

The construction of [29, 13] produces the world-volume theories describing configurations of any set of branes which can become BPS at or near the orbifold point. These are quiver gauge theories $(n_1\ n_2\ n_3)$ labelled by three integers n_i:

The theory has gauge group $\mathrm{U}(n_1) \times \mathrm{U}(n_2) \times \mathrm{U}(n_3)$ and matter content $3(n_1, \bar{n}_2) + 3(n_2, \bar{n}_3) + 3(n_3, \bar{n}_1)$. Let the chiral superfields be notated $Z^i_{a,a+1}$ with $a \in \mathbb{Z}_3$; there is a superpotential $W = \epsilon_{ijk} \operatorname{tr} Z^i_{12} Z^j_{23} Z^k_{31}$. The gauge group contains three U(1) factors, one of which acts trivially, allowing for two independent Fayet-Iliopoulos terms; these are controlled by the closed string moduli (which also have real dimension 2). There is a $\mathbb{Z}_3$ symmetry acting on the set of theories by cyclically permuting the n_i's and the associated FI terms.

The basic result [13, 30] is that the theory (1 1 1) describes a D0-brane moving on $\mathcal{M}$. The Higgs branch of its moduli space is generically a non Ricci-flat ALE space with $c_1 = 0$ and Kähler class depending on the FI terms. The $\mathbb{Z}_3$ symmetry acts on the FI terms in a way consistent with identifying it with the $\mathbb{Z}_3$ monodromy of the orbifold point.

The other theories describe different sectors in string theory. One can compute the twisted Ramond-Ramond charge and find it is non-zero for $n_i \neq n_j$, so these are branes wrapped about the exceptional cycle or "fractional branes." The elementary theories are (1 0 0) and its images under $\mathbb{Z}_3$. Since these all preserve the same supersymmetry and sum to the D0, all of their central charges are equal to 1/3; combining this information with the CFT intersection form fixes the identification of the indices n_i with the orbifold charge basis defined in section 2 [10]. In particular, (1 0 0) is the "pure" D4-brane (the state which becomes massless at the conifold point).

The definition of a bound state is a gauge theory (n_1 n_2 n_3) with a Coulomb branch corresponding to that of a single object in the Minkowski dimensions. This will be true if the unbroken gauge symmetry is precisely U(1).

The question of which combinations of n_i correspond to bound states has a priori different answers in the classical and quantum theory. We will discuss the classical theory, in which the theory (n_1 n_2 n_3) describes a bound state if it admits a gauge equivalence class of vacua (satisfying the superpotential and D-flatness conditions) which break the gauge symmetry to U(1). As we discuss in the appendix, such configurations correspond to "semistable Schur representations" of the quiver with relations.

We now claim that, with the single exception of the theory (1 1 1) (the D0), such representations of the $\mathbb{Z}_3$ quiver always come from representations of the Beilinson quiver. In other words, the $Z^i_{a,a+1} = 0$ for one $a \in \mathbb{Z}_3$ and all i. The precise relation depends on which region of moduli space we consider; let us consider the region with $\zeta_3 > 0$ and $\zeta_1 < 0$.

Looking at states with all $n_i \neq 0$ we need to solve the additional F flatness constraints: $Z^i Z^j = Z^j Z^i$ for all three pairings. Naively the expected dimension for this quiver is always negative; this is too naive because the F flatness constraints will always be redundant. For example, in the case (1 1 1) four conditions are redundant leading to the dimension 3 for the D0 moduli space.

An easy way to eliminate most of these constraints is to set $Z_{a,a+1} = 0$ and reduce to the Beilinson quiver, which we know has Schur representations. In the region of moduli space under discussion, this produces a Beilinson quiver with n_2 at the middle vertex and thus the discussion of the previous section tells us for all charge vectors whether or not Schur representations exist.

We have not proven that this exhausts the Schur representations. However, we have two reasons for believing this is so. Physically, taking all $Z_n \neq 0$ allows the brane to leave the exceptional divisor (the $\mathbb{P}^2$), and the only BPS brane we expect to do this is the D0.

A purely mathematical argument for this point might go as follows. First, if we had set $Z^2_{n,n+1} = Z^3_{n,n+1} = 0$, the single matrix $Z^1_{n,n+1}$ could only break the gauge symmetry to at least $\mathrm{U}(1)^n$ where $n = \min(n_1, n_2, n_3)$.

Then, generic solutions with $Z^2 \neq 0$ and $Z^3 \neq 0$ do not break any more gauge symmetry. This can be seen by assembling the Z^1, Z^2 and Z^3 into the original matrices of the underlying $\mathcal{N} = 4$ theory (satisfying $\gamma^{-1} Z^i \gamma = \omega Z^i$), for which the F flatness conditions are $[Z^i, Z^j] = 0$. The solutions of these are then

$$Z^2 = f^2(Z^1)\,; \qquad Z^3 = f^3(Z^1) \tag{4.1}$$

which break the same gauge symmetry as Z^1.

This leaves only the possibility of non-Beilinson representations with some $n_i = 1$. We have looked at the examples (1 1 n) and (1 2 2) in detail and found that these solutions do not break to U(1); taking larger n_2 and n_3 drives the expected dimension negative and thus seem unlikely to lead to Schur representations.

The conclusion of this section is that holomorphic objects near the orbifold point, with the exception of the D0, come from representations of the Beilinson quiver. As we saw in the previous section, each such representation is in precise correspondence to one particular large volume sheaf (it would be interesting to see this more physically, perhaps by using a D0-probe). Thus in this theory we can make a very detailed identification between many of the holomorphic objects at very distant points in Kähler moduli space. This seems to us to be striking evidence for a strong form of the "decoupling conjecture" of [16], that the holomorphic objects are independent of Kähler moduli. On the other hand, the two sets of holomorphic objects are not literally identical, but we defer discussion of this point to section 6.

5. BPS branes in the large volume limit and near the orbifold point

We are now in a position to compare the BPS spectra near these two points. Based on the behavior of the periods and an analogy to the case of pure $\mathcal{N}=2$ gauge theory [31] it was suggested in [1] (for the quintic, but the situation here is very similar) that the spectrum in the stringy regime could be substantially smaller than the large volume spectrum. In fact the spectrum near the orbifold point is strongly dependent on the direction we approach it from, and we will see in what sense this suggestion is true.

We start by completing the discussion of stable bundles on $\mathbb{P}^2$ [11]. The basic criterion to get any representation at all is that the expected dimension of the moduli space ((3.6) or (3.9)) $d \geq 0$. The regular solutions have $d > 0$ or equivalently $\Delta > 1/2$, and this part of the spectrum is approximately described by a single quadratic inequality,

$$Q_2^2 - 2Q_0Q_4 - Q_4^2 > 0\,. \tag{5.1}$$

This is however not the precise description for two reasons, which will lead to finer structure near the boundary of (5.1). First, $d = 0$ for an infinite discrete set of "exceptional bundles" with $0 < \Delta < 1/2$. Then, we need to check μ-stability for all of these bundles, using (3.7).

We proceed to quote this precise result in the next section. First, we quote the set of exceptional bundles $\mathcal{E}$. Then, one can show [11] that for bundles of given slope μ, a single exceptional bundle with slope α dominates (3.7), reducing it to a single inequality

$$\Delta \geq \delta_\alpha(\mu)\,. \tag{5.2}$$

5.1 Exceptional bundles and stable bundles on $\mathbb{P}^2$

We now describe the set $\mathcal{E}$. First, by considering tensor products with line bundles and dualization, one sees that $\mathcal{E} \cong -\mathcal{E} \cong \mathcal{E}+1$. Starting with $\mathbb{Z} \subset \mathcal{E}$, one can produce the remaining elements by the following inductive procedure. Let r_α, χ_α and Δ_α be the invariants associated to an exceptional bundle of slope α as above.

It is convenient to parameterize points in the set $\mathcal{E}$ by fractions of the form $p/2^q$. So, let $\epsilon(p/2^q)$ be an increasing function from the set of such fractions to the set $\mathcal{E}$, with $\epsilon(n) = n$ for $n \in \mathbb{Z}$. Then

$$\epsilon\left(\frac{2p+1}{2^{q+1}}\right) = \epsilon\left(\frac{p}{2^q}\right) * \epsilon\left(\frac{p+1}{2^q}\right) \tag{5.3}$$

where

$$\alpha * \beta = \frac{\alpha+\beta}{2} + \frac{\Delta_\beta - \Delta_\alpha}{3+\alpha-\beta}\,. \tag{5.4}$$

The first few examples are

$$\begin{array}{cccccc} \frac{p}{2^q} & 0 & \frac{1}{8} & \frac{1}{4} & \frac{3}{8} & \frac{1}{2} \\ \epsilon(\frac{p}{2^q}) & 0 & \frac{5}{13} & \frac{2}{5} & \frac{12}{29} & \frac{1}{2} \end{array} \tag{5.5}$$

and they rapidly accumulate to $\Delta \to 1/2$.

Using this description, it can be shown that the condition (3.7) follows from a simpler condition on Δ and μ. We define the function $\delta(\mu)$ as

$$\delta(\mu) = \sup_{\alpha\in\mathcal{E};|\mu-\alpha|<3} P(-|\alpha-\mu|) - \Delta_\alpha\,. \tag{5.6}$$

This function satisfies $1/2 < \delta(\mu) < 1$. Non-exceptional sheaves then satisfy (5.2).

It is useful to give a geometric construction of $\mathcal{E}$ and $\delta(\mu)$. Given two slopes α and β, one can plot the functions $\delta_\alpha(\mu) = P(-|\mu-\alpha|) - \Delta_\alpha$ and $\delta_\beta(\mu)$. Between α and β they are parabolas with 'acceleration' 1; the point of intersection determines a new exceptional slope γ. Δ_γ can be determined with the help of (3.6). By starting with two neighboring integers $\alpha = n$ and $\beta = n+1$, this gives a recursive description of ϵ. It can be shown that the rank r_γ of the new exceptional bundle is given by

$$r_\gamma = r_\alpha r_\beta (3+\alpha-\beta)\,. \tag{5.7}$$

Around each $\alpha = \epsilon(p/2^q) \in \mathcal{E}$ there exists a symmetric interval I_α defined by $P(-|\alpha-\mu|) - \Delta_\alpha > 1/2$. In this interval $\delta(\mu)$ can be written as

$$\delta(\mu) = \delta_\alpha(\mu) = P(-|\alpha-\mu|) - \Delta_\alpha\,. \tag{5.8}$$

The intervals shrink for increasing q and thus the function $\delta(\mu)$ has a fractal structure near $\Delta = 1/2$.

5.2 The spectrum near the orbifold point

As discussed in the appendix, the spectrum of BPS branes near the orbifold point consists of the θ-stable representations, where θ are the Fayet-Iliopoulos terms up to a possible overall shift. They are determined from the moduli [15]. As we will describe below, the spectrum near the orbifold very strongly depends on the angle by which we approach this point.

Although the $\mathbb{Z}_3$ quiver has $\mathbb{Z}_3$ symmetry, the spectrum does not, because of the non-zero Fayet-Iliopoulos terms. If we come down from the large volume limit along a particular path (we will generally consider the negative real axis as in section 2), we will obtain a particular sign for these terms, say $\zeta_3 > 0$, $\zeta_2 = 0$ and $\zeta_1 < 0$. This will pick out one of the three sets of links as naturally zero; the reductions to the Beilinson quiver which take this set non-zero will not have θ-stable representations. Thus the basic picture of the spectrum on this line is the stable representations of one of the Beilinson quivers.

Again, the basic criterion for a simple representation to exist is that the expected dimension of moduli space satisfies $d \geq 0$. Since the expected dimension formula was the

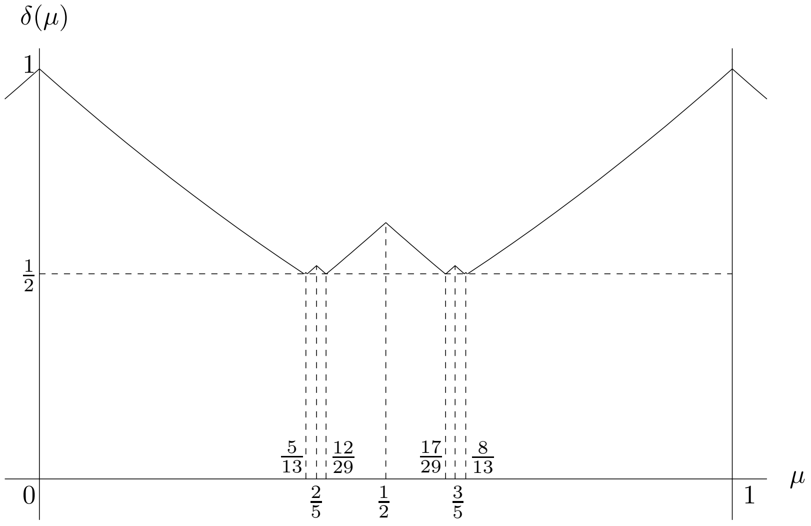

Figure 4: The function $\delta(\mu)$.

same as that for large volume bundles, this leads to exactly the same inequality (5.1) and exactly the same spectrum of exceptional representations. However there is an additional constraint, namely that all $n_i \geq 0$ or else all $n_i \leq 0$. There is clearly no $\mathbb{Z}_3$ quiver theory which can represent an object with some $n_i n_j < 0$; this would be a bound state of branes and antibranes at the orbifold point.

Although in general one can certainly obtain BPS branes as bound states of branes and antibranes, a basic claim implicit in [29] and supported by all subsequent work on this subject is that very near the orbifold point, the theories obtained by the quotient prescription describe all branes (i.e. describe all boundary states). One does not need to take bound states of branes and antibranes and would get a redundant description of existing branes if one did (this is for branes at a single fixed point; one can get new states by taking branes and antibranes at different fixed points). Thus we consider this inequality to be a correct statement about the spectrum near the orbifold point. In particular, states such as the D2 which would have gone massless there do not exist near the orbifold point.

Besides this additional inequality, the condition for stability is different. It depends on the particular line coming into the orbifold point, which determines the Fayet-Iliopoulos terms in the quiver theory, but nowhere near the orbifold point is it the same as the large volume μ-stability for all the D-branes.

One can easily check that neither the inequalities $n_i > 0$ nor the θ-stability conditions are invariant under tensoring by line bundles or equivalently $B \to B + 1$, and it is easy to see that the BPS spectrum is not either (e.g. by considering images of the fractional branes). Indeed, from the stringy point of view, such a symmetry was not to be expected, but this is already a striking difference from μ-stability.

Let us discuss some aspects of the θ-stability condition. As explained in [15], an object with charge n will decay into products including a subobject of charge n' on the line

$$\frac{\zeta \cdot n'}{e \cdot n'} = \frac{\zeta \cdot n}{e \cdot n} \tag{5.9}$$

(where e is the vector with components $e_i = 1$). The relation between the θ's we use to check θ-stability and the physical FI terms is [15]

$$\theta = \zeta - \frac{\zeta \cdot n}{e \cdot n} e \,. \tag{5.10}$$

This relation can be derived by requiring a quasi-supersymmetric vacuum for BPS branes.

In terms of the angle ϕ by which we approach the orbifold point we have $\zeta_n = r \sin(\frac{2\pi n}{3} + \phi + \pi)$. We have taken $\phi = \pi$, i.e. the negative real axis, as our standard line to the large volume limit. For orientation, we note that there are two other lines $\phi = \pi + 2\pi k/3$ which also lead to the large volume limit, but with charges differing by a $\mathbb{Z}_3$ monodromy. Similarly the lines $\phi = 2\pi k/3$ are all lines to different copies of the conifold point.

We now give for every brane satisfying $n_i n_j \geq 0$, a line leaving the orbifold point on which it is most likely to be stable, using an argument similar to the large volume argument for the necessary condition for μ-stability summarized in section 3. Specifically, we want to find ζ such that for every n' with $\theta \cdot n' < 0$, we have $\chi(n', n) \leq 0$, which is compatible with the absence of a $\mathrm{Hom}(E', E)$. If we take

$$\zeta = (\mathrm{sgn}\, n)\ \chi \cdot n$$

where χ is the matrix in (3.11), we have $\theta \cdot n' = \chi(n', n) - \chi(n, n) e \cdot n'/e \cdot n$. This works for a regular brane since it has $\chi(n, n) \leq 0$. An exceptional brane has $\chi(n, n) = 1$, and since $e \cdot n' < e \cdot n$ for a subobject, and $\chi(n', n) \in \mathbb{Z}$, this also works.

This is the analog of the necessary condition in large volume, but we have not proven its sufficiency. Nevertheless it seems to work in examples, so we will go on to assume it is sufficient in building our picture.

We next argue that each brane stable at large volume exists on a wedge near the orbifold point O containing this line. First, if a brane is stable at a point z near O, it can not be stable at the point $-z$. Thus, if we start from the line and move either clockwise or counterclockwise near O, we must encounter lines of marginal stability. One can also see from (5.9) and the definition of subobject that depending on whether we move clockwise or anticlockwise, the subobject that triggers the decay will be different.

One must look at the list of subobjects to find these lines. It may well be that as at large volume, the results will be summarized in a simple condition, but we have not studied this systematically, rather we considered examples.

The simplest example is that all branes with $n_3 > 0$ have $(0\ 0\ 1)$ as a subobject, corresponding to the need to have $\theta_3 > 0$ for a bound state. Thus they will decay at the latest when $(\zeta_3 e - \zeta) \cdot n = 0$, i.e. $\tan \phi = -(1 + 2n_1/n_2)/\sqrt{3}$. One can easily find examples in which there is no previous decay and this result implies that an infinite number of marginal stability lines come into the orbifold point.

Another possible subobject is $(0\ n_2\ n_3)$, which would trigger a decay at $\tan \phi = (1 + 2n_3/n_2)/\sqrt{3}$. Notice that this subobject is not necessarily Schur (simple), but that if so there will be a subsubobject which leads to an earlier decay. There are examples where it is Schur, giving concrete examples of decays not triggered by a fractional brane.

6. The spectrum at general points

In this section we will study the spectrum of Π-stable objects along a line (the negative real axis) connecting the large volume and orbifold points. Besides getting a more detailed picture of how the BPS spectrum varies, we will be testing whether Π-stability indeed describes sensible physics.

An object E is Π-stable if all of its subobjects E' satisfy

$$\varphi(E') < \varphi(E)\,, \tag{6.1}$$

where the grade is defined in terms of the BPS central charge $Z(E)$ as $\varphi(E) = \frac{1}{\pi}\operatorname{Im}\log Z(E)$.

This presupposes that we know all objects and their subobjects, in other words we know the category of holomorphic objects. Let us first discuss marginal stability near the large volume limit, where it is plausible to assume that the category is still the category of coherent sheaves.

6.1 Marginal stability near the large volume limit

An interesting question is whether there are lines of marginal stability arbitrarily close to the large volume limit. Using Π-stability we will argue that this is indeed the case. To see this, we expand the asymptotic form of the central charges (2.5) in terms of $t = B + iJ$, for small differences in the phase (no grading) and we obtain

$$\left(\frac{J^2}{2} + \frac{B^2}{2} - \frac{1}{8}\right)(\mu - \mu') - (\mu - B)\frac{\mathrm{ch}_2'}{r'} + (\mu' - B)\frac{\mathrm{ch}_2}{r} > 0 \tag{6.2}$$

as the modified stability condition. In the limit $J \to \infty$ this reduces to μ-stability. If $\mu = \mu'$, the criterion is for $\mu > B$

$$\frac{\mathrm{ch}_2}{r} > \frac{\mathrm{ch}_2'}{r'}\,. \tag{6.3}$$

This is known as Gieseker stability in the math literature.[4] Note however, that for $\mu < B$ we get the opposite inequality. It would be interesting to know the significance of this.

We now construct a sequence of bundles stable in the large volume limit, E_n, which by choosing sufficiently large-N will decay arbitrarily close to the large volume limit, according to (6.2). In fact, this will be true of any sequence such that all E_n have $\mathcal{O}(1)$ as a subobject, all $\mu_n > 1$, in the $n \to \infty$ limit $\mu_n \to 1$, and $ch_2(E_n) < C \ll 0$ for some C.

Now, the conditions for stable bundles are such that it is easy to find sequences F_n with the same properties except that we do not require that $\mathcal{O}(1)$ is a subobject. One can check that $\chi(F_n(-1), \mathcal{O}(1)) < 0$ using (3.8), so such bundles will always have $\mathrm{Ext}^1(F_n(-1), \mathcal{O}(1)) \neq 0$. Thus the extensions $0 \longrightarrow \mathcal{O}(1) \longrightarrow E_n \longrightarrow F_n(-1) \longrightarrow 0$ provide such a sequence.

[4]For a recent appearance of Gieseker stability in the physics literature see [32].

6.2 Comparison of the holomorphic categories

We know that at large volume, the relevant objects are coherent sheaves on $\mathbb{P}^2$, while near the orbifold point, the relevant objects are representations of the $\mathbb{Z}_3$ quiver. As we explained, these categories are extremely similar. However, they are not literally the same.

We already made the main points in the previous discussion. On the one hand, the map between Chern classes of sheaves and the integers n_i of the fractional brane basis does not preserve positivity: some sheaves have charges $n_i \geq 0$ and can be represented by the Beilinson monad, while others have mixed signs for the n_i and cannot be so represented. On the other hand, the orbifold category has at the very least a three-fold degeneracy compared to the large volume category coming from the $\mathbb{Z}_3$ symmetry and the fact that we can set any of the three links to zero to get a Beilinson quiver. Thus neither category strictly contains the other, so we cannot simply use the larger of the two along our trajectory.

The $\mathbb{Z}_3$ multiplicity of the orbifold category has no obvious analog at large volume (though see [17]), so it is not too clear how to base our discussion on the category of sheaves. On the other hand, physically it is fairly clear what we want to do to get orbifold states with mixed signs of n_i: we want to allow bound states of fractional branes and their antibranes. Although we made a point of saying that this was not necessary near the orbifold point, it is a priori quite plausible that this could lead to new BPS branes away from the orbifold point, and there are many concrete examples such as the D2 which look just like this.

Such bound states are naturally described by complexes, and thus the natural category to consider is that of complexes made from the $\mathbb{Z}_3$ quiver representations. Beilinson's theorem even provides a fairly explicit way for getting such complexes associated to particular coherent sheaves. The problem with this however is that these complexes are not at all in one-to-one correspondence with sheaves; rather, many complexes can have the same sheaf as cohomology, and most of the physical properties of the brane (stability, the moduli space, and so on) depend on which one we take. Ideally, this would not be a problem as one would find that at most one complex was stable at each point in moduli space, but this is not a priori obvious.

A mathematical construction which gives us a unique representative for each complex is to form the derived category from the original category. This construction is explained in [33 − 37] and in [17] which gives additional reasons this is useful in our string theory application. Indeed, Beilinson's theorem was originally phrased in this language: the derived categories formed from the two categories we discussed are equivalent. However, at least at present, we do not know how to use the derived category in discussions of stability, for reasons also explained in [17].

Having said this, we proceed to work with the category of complexes of $\mathbb{Z}_3$ quivers, and give examples of how some of the decays which we know must take place on the way to the orbifold point could work.

6.3 Flow of gradings

So far we have only discussed the comparison between the holomorphic objects in the large volume and orbifold regimes, but for the categories to agree, the morphisms must also agree. As is implicit in [34] and as discussed in [17], this will only be true if we take into account a spectral flow of their gradings which is determined by the flow of the brane central charges.

A simple example which is relevant for stability is the pair of branes $\mathcal{O}$ and $\mathcal{O}(-3)$. Both are stable at large volume and on the negative real axis coming into the orbifold point, but on the opposite (conifold) line $\mathcal{O}(-3)$ will decay. This is clear from its fractional brane charge (6 3 1), and in terms of θ-stability happens because (0 0 1), which is $\mathcal{O}$, is a subobject.

The confusing thing about this is that $H^0(\mathcal{O}, \mathcal{O}(-3)) = 0$, so this subobject relation is certainly not true at large volume.

However, this comes about because in the large volume limit, as follows from Serre duality, one has

$$\dim H^2(\mathcal{O}, \mathcal{O}(-3)) = 1 \tag{6.4}$$

and the flow of the gradings turns this into the Hom of the orbifold point.

A useful notation is to indicate the gradings of both branes and morphisms by square brackets, so the homomorphism we described becomes

$$\dim \mathrm{Hom}(\mathcal{O}, \mathcal{O}(-3)[2]) = 1\,. \tag{6.5}$$

From section 2 one sees that as one goes from large volume to orbifold, the two periods go around the origin on different sides. This means that the relative grading is shifted by 2, and the result (6.5) turns into

$$\dim \mathrm{Hom}(\mathcal{O}, \mathcal{O}(-3)) = 1 \tag{6.6}$$

for the natural (zero grading) objects at the orbifold point.

6.4 Another technical result on Π-stability

In practice it is much easier to see whether there are homomorphisms between objects, than to see whether they are injective. For μ-stability one circumvents this difficulty by arguing that if there exists a $\phi \in \mathrm{Hom}(E', E)$ from a *stable* object with $\mu(E') \geq \mu(E)$, then E will be destabilized, either by E' or by the preimage of ϕ, which will be an object E'' with an exact sequence

$$0 \longrightarrow E''' \longrightarrow E' \longrightarrow E'' \longrightarrow 0\,. \tag{6.7}$$

Because E' is stable, $\mu(E'') > \mu(E')$. This follows from $\mu(E''') < \mu(E')$, $r''' + r'' = r'$ and $c_1''' + c_1'' = c_1'$.

The same is true of Π-stability, if the differences between the gradings of any two objects are less than 1. This follows from the same argument but deriving the relation $\phi(E') < \phi(E'')$ from $Z' = Z'' + Z'''$ and the convexity of $\phi = \mathrm{Im} \log Z$ under this assumption. It may be true more generally, but we have no argument for this.

6.5 The D2-brane is born, and other examples

In this section we want to motivate the line of marginal stability for the D2-brane. The orbifold charges of the D2-brane are $(1\ 0\ -1)$. This brane can obviously not exist at the orbifold point, but it exists in the large volume. Looking at the evolution of the periods of the fractional branes in figure 1 one finds that at a point P on the negative ξ-axis the periods Π_1 and Π_3 become antiparallel. This is the obvious proposal for the location of a line of marginal stability for the D2-brane, at which it decays into $(1\ 0\ 0)$ and $(0\ 0\ -1)$, in other words $\bar{\mathcal{O}}(-1)$ and $\mathcal{O}$.

For Π-stability to describe this process, it must be that one of these objects is a subobject of the D2. We can describe the D2 by a sheaf supported on a two-cycle, $\mathcal{O}_\Sigma$, defined by the exact sequence

$$0 \longrightarrow \mathcal{O}_i \longrightarrow \mathcal{O} \xrightarrow{\phi} \mathcal{O}_\Sigma \longrightarrow 0$$

where $\mathcal{O}_i$ is the ideal sheaf of Σ, i.e. functions vanishing on this divisor. Clearly the map ϕ is our candidate homomorphism.

An interesting point about this is that the large volume gradings between branes of complex dimensions differing by an odd integer are naturally half integral (from $\frac{1}{\pi}\operatorname{Im}\log(B+iV)^p$). We assign ϕ in $\operatorname{Hom}(\mathcal{O}[0], \mathcal{O}_\Sigma[1/2])$, which will flow to zero degree at the point P.

One also might not have thought that ϕ was injective from the large volume definition. As all of the 2B's will decay into combinations of 4B's, this is a very general problem. The result of the previous section tells us that we could still have a decay, but this would go into an object different from $\mathcal{O}$ (the E''' above), which seems rather implausible without further evidence.

The Hom is injective on global sections, so perhaps this is the correct definition. We can also see that there should be an injective Hom (and get some ideas for a more general definition of the category) by constructing a complex of $\mathbb{Z}_3$ quivers which reproduces the 2B. This can be done in various ways by taking two trivial (single node) quivers for the two constituents and adding brane-antibrane pairs. The simplest way is to take $(1\ 0\ 1)$ and $(2\ 0\ 0)$ to represent the antibrane and brane respectively, use the third link of the $\mathbb{Z}_3$ quiver (the one which was zero in the Beilinson quiver) to make a simple object out of $(1\ 0\ 1)$, and then postulate two "tachyon" links of opposite orientations between brane and antibrane which are turned on to make the bound state. The moduli space of the resulting complex is indeed $\mathbb{P}^2$ as is correct for the 2B, and one sees that there is an injective Hom from $(0\ 0\ 1)$ into this complex.

Although this construction is not unique, it shows that there is a definition of the 2B consistent with the general picture we are suggesting.

Another interesting example is the bundle $\Omega^1(2) = T(-1)$, an exceptional rank 2 bundle. Its only subsheaves are $\Omega^1(1-n)$ and $\mathcal{O}(-n)$ for $n \geq 0$. From the Euler sequence one can determine its charges to be $(1\ 0\ -3)$, and according to Π-stability it is destabilized by $\mathcal{O}$ at the same point P as the D2-brane, where it decays into $3\ \mathcal{O} + \bar{\mathcal{O}}(-1)$.

Since there is a $\mathbb{Z}_2$ symmetry of the moduli space, the reflection on the real ξ-axis, which acts on the spectrum by exchanging n_1 and n_2 ($\mu \to -1-\mu$), the brane $(1\ 0\ -3)$, i.e. Ω^1 has the same line of marginal stability, but there is no subsheaf destabilizing it. By analogy to the decay above, the obvious candidate for the subobject triggering the decay is $\bar{\mathcal{O}}$.

In fact, a homomorphism in $H^1(\mathcal{O}, \Omega^1)$ can explain this decay, taking into account the flow of gradings. These must exist because $\chi(\mathcal{O}, \Omega^1) = -2$.

6.6 The conifold point and the general picture

The conifold point would be a particularly interesting test of the general formalism as the issue of how to treat large variations of the gradings comes to the fore here. This is simply because a closed loop around the conifold point increases the grading of $\mathcal{O}$ (the brane which becomes massless there) by two.

We do not have a solvable perturbative string theory realization of this region of the moduli space, but the basic physics is quite clear from the known relation to Seiberg-Witten theory [38] and the results so far. On the large volume side of the conifold point, there is no reason to expect the spectrum to be drastically different from the large volume spectrum. On the other hand, we know that if we encircle it, we would produce bound states with arbitrarily large D4 number which do not exist in the large volume spectrum; therefore there must be lines of marginal stability through the conifold point on which all of these decay. The vanishing of the D4 central charge there means that any state containing D4 charge (in some basis) will indeed see candidate marginal stability lines on which it will decay to the D4 and other products.

Since we have two other charges in the picture, not just the one of gauge theory, we need additional information to decide which states exist on the other side. We will do this by outlining a global picture of the marginal stability lines we have already found.

Now, making a truly global picture would require a good description of the "Teichmüller space," the universal cover of the thrice-punctured moduli space, or at least some cover on which periods are single-valued. Since this is a one dimensional moduli space with a natural metric of negative curvature, one might imagine that it could be described as some quotient of the upper half plane.

Such a description is not known. One serious complication compared to the familiar examples such as gauge theory where this is possible, is that the period map, which in this example takes the Teichmüller space into $\mathbb{C}^2$ (this is local mirror symmetry so $\Pi_{D0} = 1$), behaves badly on the boundary (it seems to map it into a fractal in $\mathbb{C}^2$).

Not having this, we restrict attention to the triple cover of moduli space near one orbifold point O, parameterized as in the introduction. Near the orbifold point, this cover will have three conifold points C_i where the ith fractional brane B_i becomes massless. There are also three large volume points LV_i opposite the conifold points. The large volume point that we normally consider is LV_3. The monodromies associated with these points are described by branch cuts in the periods which leave this region, and the full

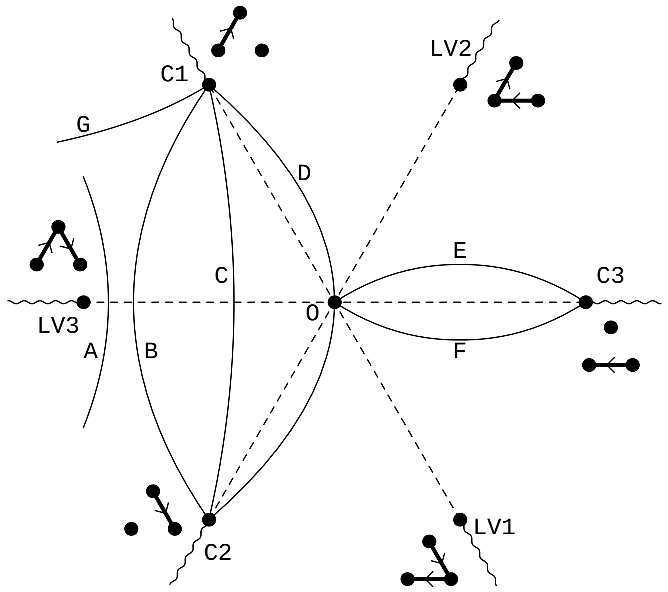

Figure 5: The structure of the Teichmüller space.

Teichmüller space will be covered by copies of this region in which the role of the fractional branes is played by monodromy images of our three fractional branes.[5]

We can now trace the marginal stability lines we found near the orbifold point and intersecting the negative real axis out into this region. As we discussed, the natural endpoint for a line describing a decay into a fractional brane B_i is the conifold point C_i. We give a map in figure 5 with various lines which we proceed to explain.

We can distinguish two basic types of marginal stability lines at this point. On the one hand, we found lines coming out of the orbifold point with decays triggered by fractional branes. The simplest picture is that these head along a (fairly direct) path to the appropriate conifold point, as in lines D, E and F on the figure. We will refer to these as "CO" lines.

We also found decay lines for branes such as the D2 which do not exist near the orbifold point. Now the D2 and the similar objects $(n_2\ 0\ n_1)$ will decay into a fractional brane and a different fractional anti-brane. The simplest picture is that such a line connects the two associated conifold points, as do lines B and C on the figure. We will refer to this as a "CC" line.

In general, both types of line will also exist for decays into other exceptional branes, and should lead to the conifold points in other regions of the Teichmüller space, as do lines A and G. There might also be decays purely into regular branes. Since monodromies do not in general preserve the dimension formula (3.13), one might wonder if these could also be associated with monodromy images of exceptional branes. The strong decoupling hypothesis [17] requires all such monodromies which change the dimension of moduli space of the holomorphic objects to cross lines of marginal stability, and would disallow this; then these lines would presumably be "OO" lines between different orbifold points.

[5]This part of the story is related to the theory of helices. [39]

Note that just because a marginal stability line crosses the negative real axis does not imply that it is a CC line. For example, by tensoring the bundles we discussed as examples of decays near the large volume limit with appropriate line bundles, one can get examples of decays on the negative real axis of objects with all n_i positive. These must exist near the orbifold point so these are in fact CO lines.

There are a few more general considerations we can use to constrain the picture. First, there appear to be no marginal stability lines from a singularity to itself. For the orbifold this is because (as discussed earlier) the two marginal stability lines for a given object are triggered by different subobjects, while for the conifold this should follow because we do not expect an object to decay both into a brane and the same antibrane.

Two marginal stability lines involving the same object cannot cross in this problem, except at orbifold points, because such a crossing requires at least three periods to align. In particular the CO lines triggered by the same fractional brane cannot cross, which makes the hypothesis that they take fairly simple paths well-motivated.

The symmetry of the problem about the negative real axis, and other symmetries, should be useful. We leave the problem of finding the complete picture however to future work.

As for the question of how small the spectrum can get, these considerations suggest that the smallest spectrum is already attained near the orbifold point and near its line to a conifold point. The general character of the spectrum here is probably illustrated by the decay to (0 0 1) which as we discussed takes place on a line with angle $\tan\phi = -(1+2n_1/n_2)/\sqrt{3}$. If we start on the line to LV_3 and decrease ϕ to $2\pi/3$, branes with $n_1 < n_2$ will decay before this line, while branes with $n_1 > n_2$ will have marginal stability lines such as D (assuming they decay into $\mathcal{O}$), and survive on this line.

Going farther will produce further decays of this type, and in this sense the spectrum can become arbitrarily small. On the other hand, if we go too far, we will pick up objects naturally associated with LV_2, and the spectrum will grow again.

If we take the line $\phi = 2\pi/3$ as representative, the overall picture is one in which almost all bound states with the D4 do decay as we go around the conifold, but along any given line, a large subset (with three adjustable charges satisfying inequalities) do exist.

We finally comment on the role of the D0-brane, which naively one might think would play a central role in the story. Although the D0 appears to exist as a BPS brane everywhere except at the orbifold point itself in this model, this is to some extent a simplification of local mirror symmetry (its conjugate, the D6, has infinite tension in this limit). The D0 is clearly not the simplest object near the orbifold point and is not naturally a subobject of other branes at large volume either, since one does not have natural maps to sheaves of higher dimension. Thus the results here support the idea that it (and all non-exceptional branes) is best thought of as a bound state of simpler objects, the exceptional branes.

7. Bound states on the quintic

The approach we discussed can surely be extended to other orbifold singularities and non-compact CY's (some relevant mathematical work is [40]). But can these ideas describe branes and stability on compact CY's?

A natural way to try to describe bundles on varieties in projective space is by constructing bundles on projective space and restricting them to the subvariety. Of course not all bundles can be so obtained, but this is already a much more general class than has been available in any of the constructions commonly used by physicists (such as [41]).

Although the complete analysis of stable bundles is not known for $\mathbb{P}^d$ with $d > 2$, many of the ingredients we described are known. In particular, Beilinson's construction of holomorphic bundles generalizes directly to $\mathbb{P}^d$. Given a bundle E on P^d, the data $H^p(E(-q))$ and natural maps between these spaces defines a spectral sequence whose cohomology is E, and in favorable cases this again reduces to a complex whose cohomology is E. The complex is the obvious generalization of (3.16):

$$0 \to \mathbb{C}^{n_d} \otimes \mathcal{O}(-1) \xrightarrow{\hat{X}^{d-1}} \mathbb{C}^{n_{d-1}} \otimes \Omega^{d-1}(d-1) \xrightarrow{\hat{X}^{d-2}} \cdots \longrightarrow \mathbb{C}^{n_2} \otimes \Omega^1(1) \xrightarrow{\hat{X}^0} \mathbb{C}^{n_1} \otimes \mathcal{O} \to 0 \quad (7.1)$$

with $X^n_{[i} X^{n-1}_{j]} = 0$. In quiver terms, we have $d+1$ nodes in a line, with $d+1$ arrows between successive nodes.

It turns out that there is a very direct analog of this structure implicit in the results of [16] describing rational B branes in the $(3)^5$ Gepner model [42]. These branes are labeled by quantum numbers $0 \leq L_i \leq 1$ in each minimal model factor and an overall $0 \leq M < 10$, which is even for branes and odd for antibranes.

Since the elementary bundles $\Omega^n(n)$ of this construction are all rigid, their natural counterparts are the $L = 0$ branes, of which we take the five branes $M = 2i$. We thus consider an $\mathcal{N} = 1$ quiver gauge theory defined using any number n_i of each of these branes, and chiral multiplets containing massless fermions stretched between pairs of branes. This spectrum can be read off from the results of [42, 16]: it is a $\prod \mathrm{U}(n_i)$ theory with five multiplets $X^a_{i,i+1}$ in each $(n_i, \bar{n}_{i+1})$ and ten multiplets $Y^{[ab]}_{i,i-2}$ in each $(n_i, \bar{n}_{i-2})$. (Here $1 \leq a, b \leq 5$ are global quantum numbers).

We immediately notice that the links X are in exactly the same relation to the Beilinson complex for $\mathbb{P}^4$ as the links of the $\mathbb{Z}_3$ orbifold model were to the monad for $\mathbb{P}^2$, if we set $X_{5,1} = 0$. The relations suggest the following ansatz for the superpotential:[6]

$$W = \sum_{i,a,b} \operatorname{tr} X^a_{i,i+1} X^b_{i+1,i+2} Y^{ab}_{i+2,i}\,,$$

F flat configurations of this gauge theory with $Y = X_{5,1} = 0$ are exactly representations of the Beilinson quiver.

Furthermore, according to CFT, the bosons in the X multiplets are tachyonic, so by condensing them one can find bound states of branes. The natural conjecture is that these bound states are the Gepner point images of a large set of bundles on the large volume quintic, those obtained as the cohomology of the complex (7.1).

In particular, this identification predicts that the bundles corresponding to $L = 0$ branes are the bundles $\Omega^n(n)$. One of the $L = 0$ branes was found in [16] to be the D6-brane $\mathcal{O}$; it is easy to check that its images under the $\mathbb{Z}_5$ monodromy of the Gepner point have exactly the predicted Chern classes.

[6]This term in the superpotential can also be seen in CFT. [43]

We have furthermore identified some of the simpler bound states with $L > 0$ branes. The simplest case is the bound state (1 1 0 0 0) which according to the quiver gauge theory has moduli space dimension 4. This is exactly the charge of a rational boundary state with $\sum L_i = 1$ and indeed the number of marginal operators for this boundary state is 4. We made the same check for the bound state (1 2 1 0 0) which is the charge of a $\sum L_i = 2$ brane, and match the correct number of marginal operators 11.

These results begin to answer one of the main questions raised in [16] and related work — the large volume identification of the B rational boundary states. Of course for these to actually exist as BPS states at large volume, they must be stable bundles. There is a further correspondence in that the brane predicted to decay in [1] is in fact a bound state which would not exist at large volume (it uses the link $X_{5,1}$ which is not present in the Beilinson complex).

8. Conclusions

We studied the spectrum of BPS branes on the non-compact Calabi-Yau space $\mathcal{O}_{\mathbb{P}^2}(-3)$, and gave detailed results for the spectrum near the large volume limit and near the orbifold point, and a qualitative picture elsewhere.

We see this work as shedding light on at least two aspects of string physics on Calabi-Yaus: besides describing a highly non-trivial example of an $\mathcal{N} = 2$, $d = 4$ theory, we can also interpret our results as providing a picture of the set of possible gauge bundles which could be used in type I and heterotic string compactifications, and suggesting new ways to analyze this aspect of $\mathcal{N} = 1$ compactification. Let us discuss these points in turn.

There have been many studies of the spectrum of BPS states in $\mathcal{N} = 2$ theories and marginal stability, mostly on systems with a fairly simple spectrum, supersymmetric gauge theories and the corresponding limits of string theory compactification. In these theories the weak coupling spectrum is that of dyons with highly constrained magnetic charge but fairly general electric charge. One or a few lines of marginal stability separate the weak and strong coupling regimes, and in the latter the spectrum is drastically reduced, to a finite number of states, all of which become massless at singular points in moduli space.

The picture here is broadly similar but with noteworthy differences. The spectrum in the large volume limit is roughly characterized as those charge vectors satisfying a quadratic inequality guaranteeing that the moduli space dimension is non-negative, but there is a subtle fine structure of the boundary of this region in which certain rigid exceptional bundles exist and destabilize other regular bundles. As one moves to the orbifold limit, additional inequalities come into play and reduce the spectrum. The spectrum becomes even smaller between the orbifold and conifold points but is still infinite; it appears to never reduce only to states which can become massless.

Our approach was to use a generalization of the idea of stability which governs the existence of solutions of the hermitian Yang-Mills equations and thus would determine the spectrum of BPS branes in the large volume limit. As discussed in [15] and [17], this approach divides the problem into that of studying the category of holomorphic brane configurations and then finding the stable objects in this category. The approach is proven near the orbifold point and is very well motivated near the large volume limit, where it only assumes that mirror symmetry works.

Quite strikingly, the category of holomorphic brane configurations is almost the same in these two limits: the orbifold theory essentially contains Beilinson's general construction of holomorphic bundles on $\mathbb{P}^2$. We consider this evidence for a strong form of the "decoupling conjecture" of [16], that the holomorphic category is largely independent of Kähler moduli. This similarity allows us to get at least partial results on lines connecting these two limits. It suggests that direct application of mathematical results on this category and the related derived category of sheaves such as those of [39, 44] will allow pushing this analysis to the entire moduli space. It would be particularly interesting to study the conifold point using these ideas.

Of the various alternate approaches one might take to this problem, we should mention that of [45], in which special lagrangian cycles on the mirror are reduced to string networks in F theory. At present this would seem to be the most promising way to get results from the A picture, and it will be interesting to see if this approach can reproduce the very detailed structure we saw here.

We discussed the classical theory of these branes, but let us make a few remarks about the quantum BPS particles in IIa string theory on this background. These will be vacua of the quantum mechanical theories obtained by dimensional reductions of the brane world-volume theories we derived. Typically, such vacua correspond to cohomology classes of the moduli space. One very general result which follows from this identification is that typically (though not always), crossing a line of marginal stability completely eliminates the moduli space, which means that all the quantum branes of that charge will decay on that line. More specific results might be obtained by using our explicit construction of the classical moduli space as a symplectic quotient of the moduli space of quiver representations. Many partial results exist on this spectrum for large volume $\mathbb{P}^2$, in particular [46, 47]. These results should also further the study of BPS algebras [22].

We turn to the $3+1$ case. The study of gauge bundles on Calabi-Yaus has long been a central part of superstring physics. At least conceptually, the direct way to study $\mathcal{N}=1$ compactification and $\mathcal{N}=1$ duality would be to have a list of Calabi-Yaus (quite a good list of these is known, and this was very helpful in studying $\mathcal{N}=2$ duality) and then a list of all sheaves on them. Not only do we not have such a list, there is no picture we are aware of in the physics literature which gives any overall sense of the possibilities. Even very general questions such as in what range of Chern classes do stable sheaves generically exist or generically do not exist go unaddressed.

We hope that the present work will serve to illustrate what such a picture might look like, and believe that the ideas and techniques we discussed will apply to branes in very general Calabi-Yau compactifications. We even gave the beginnings of this for the quintic. A detailed study of this and other examples and the precise relation to large volume bundles is under way; recently in [48] we have found how to generalize the results of section 7 to a large subset of the Gepner models.

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The most direct way to use such results to study $\mathcal{N} = 1$ compactifications is to choose a brane configuration and do an orientifold projection, leading to perturbative type I models satisfying the known constraints such as tadpole cancellation. Indeed, the explicitness of the supersymmetric gauge theory description in the $\mathbb{C}^3/\mathbb{Z}_3$ and $T^6/\mathbb{Z}_3$ examples has allowed much work to be done, starting with [49], including [50] and many others.

Although from this point of view one might consider the present work as one more in a long list, the point at which we feel we have made significant progress is to provide a clear algebraic geometric interpretation of the brane configurations as specific bundles on the resolved orbifold singularity, and explain how this geometric interpretation and other relevant mathematics can be used to organize and solve the problem of classifying supersymmetric vacua. We furthermore feel that having the geometric interpretation will be a crucial element in the study of superstring dualities involving these examples.

Clearly having such explicit results for compact Calabi-Yaus would allow a much more complete study of $\mathcal{N} = 1$ compactification. This might seem rather optimistic for a problem with such a long history; time will tell. Let us conclude by listing the elements of the emerging picture of D-branes on Calabi-Yaus which may provide advantages over other approaches.

- Decoupling of complex and Kähler structure is a significant a priori simplification and appears to allow detailed comparison between constructions in different limits of Kähler moduli space.

- By starting with the classical limit, we can work with configurations which do not satisfy tadpole or anomaly cancellation constraints. This allows building bundles out of simpler constituents.

- Our results strongly suggest that branes in the stringy regime are best thought of as bound states of a finite set of rigid exceptional branes. Indeed, identifying an appropriate basis of bundles in terms of which the others are simple bound states appears to be a central theme in the mathematics (e.g. see [51]). For Fano varieties (such as $\mathbb{P}^2$) one has a precise idea of what one is looking for, an exceptional collection. Such collections do not exist for Calabi-Yaus, but we believe that as in our quintic example, the subset of rigid rational boundary states will provide the appropriate generalization of this idea.

Finally, let us state what we feel is the broadest lesson of this work. In seeing how the various elements of the problem of classifying bundles on $\mathbb{P}^2$ translate into the language of supersymmetric field theory, we realize that doing this translation is actually much easier than classifying the bundles. In the case of flat space, where the translation is provided by the ADHM construction, this has been appreciated by physicists for some time now [52].

Although classification on a Calabi-Yau is much harder, this does not mean that the translation need be so much harder. The hard parts of the classification, determining holomorphic moduli spaces and stability, get translated into aspects of the problem of finding supersymmetric vacua of the resulting theories, and the greater difficulty of solving them is essentially for the usual reasons that these problems can be hard in $\mathcal{N} = 1$ supersymmetric theory.

If such translations are generally possible, a part of the problem of string compactification that many physicists (at least the authors) have found rather mysterious and intimidating, will turn out to be not so different from the more familiar parts.

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A. Quivers and their representations

The theory of quivers is a framework encompassing a wide variety of problems in linear algebra and representation theory; a couple of nice references on the subject are [53, 54]. Some relations between quiver theory and gauge theories have been discussed recently in [55].

A *quiver* Q is a directed graph; it can be defined by a set of vertices V, a set of arrows A connecting vertices, and two functions i and t from A to V specifying the initial and final vertex for each arrow.

An *path* in the quiver has the obvious definition (the arrows must preserve orientation). The *path algebra* is the algebra generated by the paths, with multiplication given by concatenation. Perhaps the simplest mathematical definition of "quiver theory" is that it is the representation theory of such algebras.

A.1 Gauge theory and representations of quivers

Associated to a quiver Q there are a set of gauge theories $\mathcal{Q}(\vec{n})$ with gauge group $G = \prod_{v \in V} \mathrm{U}(n_v)$ and bifundamental matter. We will consider $\mathcal{N} = 1$, $d = 4$ supersymmetric theories, with a chiral multiplet x_a associated to each arrow $a \in A$, in the $(n_{ta}, \bar{n}_{ia})$ representation of G.

A point in the configuration space of this theory then corresponds to a *representation* R of the quiver Q. This is a collection of finite dimensional vector spaces $R(v)$, one for each vertex $v \in V$, and a collection of linear maps between these spaces $X_R(a)$, one for each arrow $a \in A$. The vector $\vec{n}$ with components $n_v = \dim R(v)$ is referred to as the *dimension vector* $\dim R$ of the representation.

In supersymmetric gauge theory one also makes use of the complexified gauge group CG which would be $\prod_{v \in V} GL(\dim R(v))$. It turns out to be useful to phrase the relation of gauge equivalence between configurations in a way that does not require inverting group elements. Thus we write a "gauge transformation" ϕ as a set of linear transformations $\phi(v) : R'(v) \to R(v)$ for which

$$X_R(a)\phi(ta) = \phi(ia)X_{R'}(a) \tag{A.1}$$

for all a. If $\phi \in \mathrm{U}(R)$ this is a true gauge equivalence.

This can be further generalized to a *homomorphism* of representations $\phi : R \to S$: this is a set of linear maps

$$\phi(v) : R(v) \to S(v) \tag{A.2}$$

satisfying

$$X_R(a)\phi(ta) = \phi(ia)X_S(a)\,. \tag{A.3}$$

Since we do not require invertibility, these form a group under addition, called $\mathrm{Hom}(R, S)$. If $R \cong S$ this group is the *endomorphism* group End R. Invertible (in the usual matrix sense) endomorphisms are elements of CG and thus the dimension of the endomorphism group is also the dimension of the the subgroup of the gauge group left unbroken by the configuration X_R. If there is an injective homomorphism from R to S we say that R is a *subrepresentation* of S. When the generic representation of dimension vector α has a subrepresentation with dimension vector β we say that β is a *subvector* of α.

We will say that a representation is *indecomposable* if it can not be written as a direct sum of smaller representations, while we say that a representation R is *Schur* (or simple), if End $R \cong \mathbb{C}$. Note that unlike the more familiar theories of finite groups and compact Lie groups, we need to distinguish these notions: although a Schur representation is indecomposible, the converse is not always true. The Schur representations are of particular relevance for us, since the breaking of the gauge group down to U(1) signals the existence of a bound state.

We note in passing that the quiver representations are objects in an *abelian category* with the homomorphisms as just defined. Many of the further definitions and theorems are usually discussed in this more general context.

A.2 Supersymmetric vacua of gauge theory

In the gauge theory application, a primary problem is to find the space of vacua of the theory. In the case of supersymmetric classical vacua of $\mathcal{N} = 1$ gauge theory, this means gauge equivalence classes of configurations which satisfy the "D and F flatness conditions." We require two further pieces of information about the theory to formulate these.

The first is the superpotential W, a holomorphic gauge invariant function of the chiral fields. The F-flatness conditions are then $\partial W/\partial x_a = 0$ for all a. Let us call the variety defined by these constraints the "F-reduced configuration space."

The second is qualitative information about the kinetic term; we will assume that it is non-singular. In fact we will choose $K = \sum_a \mathrm{tr}\, x_a x_a^+$, but the qualitative features of the result do not depend on this choice. The D-flatness conditions then depend on $|V| - 1$ real parameters (the "Fayet-Iliopoulos terms") and are

$$\sum_{ha=v} X_a^+ X_a - \sum_{ta=v} X_a X_a^+ = \zeta_v\,. \tag{A.4}$$

As is well-known these constraints modulo gauge equivalence are an instance of the symplectic quotient construction. Thus the moduli space of supersymmetric vacua is the symplectic quotient of the F-reduced configuration space by G.

There is a further relation between this quotient and a different definition of quotient provided by geometric invariant theory. (This is fairly well known to physicists in the case $\zeta = 0$, e.g. see [56]). The GIT quotient is defined in terms of the set of orbits of the complexified gauge group CG, acting as $x \to g^{-1}xg$. This is a space which looks roughly like the symplectic quotient — in particular, it has the same complex dimension. Furthermore, each orbit contains at most one solution of the D-flatness conditions (for some FI terms). However it turns out to be larger — some of the orbits do not contain any solution of the D-flatness conditions.

Very generally, geometric invariant theory provides a concept of *stable* orbits which are precisely those which correspond to points in the symplectic quotient, i.e. those containing a solution of the D-flatness condition. The basic idea will be that stable orbits are those which are closed under a suitably extended form of the group action. This means that if we try to find the solution by minimizing a potential (the usual D^2, say) over the orbit, the minimum must be attained. Conversely, orbits which are not closed will not contain this minimum.

In the context of quivers (or abelian categories), an appropriate definition of stability is *θ-stability*, formulated by King [14]. Let θ be a vector indexed by V; then R is *θ-semistable* if $\theta \cdot \dim R = 0$ and every subobject R' (i.e. one for which there exists an injective homomorphism $\phi : R' \to R$) satisfies $\theta \cdot \dim R' \geq 0$. Furthermore, R is *θ-stable* if the only subobjects with $\theta \cdot \dim R' = 0$ are R and 0.

The theorem is then that R is a solution of the D-flatness conditions for ζ precisely when it is a direct sum of θ-stable representations with $\theta = \zeta$.

This is a useful criterion because it translates the problem of solving the D-flatness conditions into a purely algebraic question, that of classifying possible subobjects, which can be studied inductively. Furthermore it tells us how the existence of solutions depends on the FI terms, as we discuss in examples in the main text.

We now consider the combined problem of D and F-flatness conditions. Although a theory with general superpotential does not correspond to a problem studied in quiver theory, superpotentials which can be written as a single trace produce *quivers with relations*. Each of the matrix equations

$$\frac{\partial W}{\partial X^a} = 0$$

can be thought of as defining a weighted sum over oriented paths in the quiver; equating the sum to zero defines a relation in the path algebra.

The example of immediate interest for us is the superpotential

$$W = \epsilon_{ijk} \operatorname{tr} X^i_{1,2} X^j_{2,3} X^k_{3,1} \tag{A.5}$$

for which the F-flatness conditions $W' = 0$ are equivalent to requiring a set of commutation relations

$$X^i_{n,n+1} X^j_{n+1,n+2} = X^j_{n,n+1} X^i_{n+1,n+2} \,. \tag{A.6}$$

As pointed out in [14], the discussion of θ-stability works the same way in the presence of relations. This is because if an object satisfies the relations, its subobjects will also satisfy them.

A.3 Roots of quivers

The dimension vectors of representations of a quiver with n nodes live in $\mathbb{N}^n$. We introduce now many concepts similar to those of the theory of Lie algebras. In this subsection we consider only quivers without relations.

Given the dimension vectors α and β of two representations, we can define the *Euler form*

$$\langle\alpha,\beta\rangle = \sum_v \alpha_v\beta_v - \sum_a \alpha_{ia}\beta_{ta} \tag{A.7}$$

and the *Cartan matrix* of a quiver as the symmetrization of the Euler form

$$(\alpha,\beta) = \alpha^T\cdot C\cdot\beta = \langle\alpha,\beta\rangle + \langle\beta,\alpha\rangle\,. \tag{A.8}$$

The expected dimension of the moduli space of a quiver representation (and thus of the gauge theory) is

$$d(\alpha) = 1 - \frac{1}{2}\alpha^T\cdot C\cdot\alpha\,. \tag{A.9}$$

If $d\geq 0$, we expect to find such objects, while if $d<0$ we expect not to. These expectations are made precise in a theorem of Kac [57], which we proceed to quote.

For each node of the quiver, consider the n-vector $e_i = (0,0,\ldots,1,\ldots,0)$ with the 1 in the i'th position. Let us call these n-vectors the *simple roots* of the quiver, and denote the set of them by Π. For each simple root we can introduce a Weyl reflection acting on an arbitrary vector α:

$$r_i(\alpha) = \alpha - 2\frac{(\alpha,e_i)}{(e_i,e_i)}e_i\,. \tag{A.10}$$

The *fundamental region* is defined as the set of vectors α satisfying $r_i(\alpha)\geq\alpha$ for all i, and whose *support* (i.e. the subset of nodes for which $\alpha_i\neq 0$) is connected.

The group $\mathcal{W}$ generated by the Weyl reflections is called the *Weyl group* of the quiver. The *real roots* of a quiver are defined as the set of vectors obtained by the action of the Weyl group on the simple roots.

$$\Delta^{re} = \mathcal{W}(\Pi)\,. \tag{A.11}$$

The *imaginary roots* of the quiver are the set of vectors obtained by the action of the Weyl group on the vectors of the fundamental region.

$$\Delta^{im} = \mathcal{W}(F)\,. \tag{A.12}$$

The root system of the quiver is given by the real roots plus the imaginary roots. All real roots satisfy $\alpha\cdot C\cdot\alpha = 2$ and all the imaginary roots satisfy $\alpha\cdot C\cdot\alpha\leq 0$. We define the subset of the roots which can be written as $\alpha=\sum_i k_i\alpha_i$ with $k_i\geq 0$ to be the *positive roots*.

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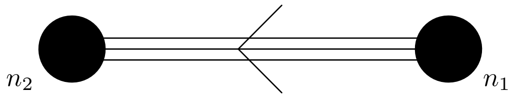

Figure 6: The Kronecker quiver.

For quivers without relations, Kac proved in [57] the following fundamental theorem: the dimension vectors of indecomposable representations are the positive roots of the quiver. The real roots correspond to rigid representations (i.e. with no moduli), while the imaginary roots correspond to representations with moduli space dimension greater than or equal to the expected dimension (A.9). The possibility of a greater actual dimension in this case is because there can be additional unbroken gauge symmetry; conversely for Schur roots (roots of Schur representations), the true dimension will be equal to the expected dimension.

Finally, we quote a result useful in finding bound states. For a quiver without relations, $\mathrm{Ext}^i(V,W) = 0$ for $i \geq 2$, and the Euler form gives the relative Euler character of two representations V, W with dimension vectors α, β:

$$\langle \alpha, \beta \rangle = \dim \mathrm{Hom}(\alpha, \beta) - \dim \mathrm{Ext}(\alpha, \beta) = \chi(\alpha, \beta) \tag{A.13}$$

which can be derived by considering the following exact sequence

$$0 \to \mathrm{Hom}(V,W) \to \sum_v \mathrm{Hom}(V_v, W_v) \to \sum_a \mathrm{Hom}(V_{ta}, W_{ha}) \to \mathrm{Ext}(V,W) \to 0\,. \tag{A.14}$$

A.4 An example: the generalized Kronecker quiver

As an illustration of the definitions we have introduced in this appendix, we study in detail the following quiver

The Euler form for this quiver is given by

$$\langle \alpha, \beta \rangle = (\alpha_1\ \alpha_2) \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}. \tag{A.15}$$

Our goal is to find the Schur roots. The first and easiest step is to find the indecomposable roots, which by Kac theorem are the positive roots of this quiver. The fundamental roots are just (0 1) and (1 0). The respective Weyl reflections are

$$r_1 = \begin{pmatrix} -1 & 3 \\ 0 & 1 \end{pmatrix}, \tag{A.16}$$

$$r_2 = \begin{pmatrix} 1 & 0 \\ 3 & -1 \end{pmatrix}. \tag{A.17}$$

The fundamental region consists of the vectors v for which $r_i(v) \geq v$, so it is the interior of the cone spanned by the vectors $u_1 = (2\ 3)$ and $u_2 = (3\ 2)$. The imaginary roots are those that can be brought to the fundamental region through Weyl reflections. They correspond to the interior of the cone spanned by the null vectors $v_{1,2} = (1, \frac{3 \pm \sqrt{5}}{2})$. Note that for imaginary roots the dimension of their moduli space is $d > 0$.

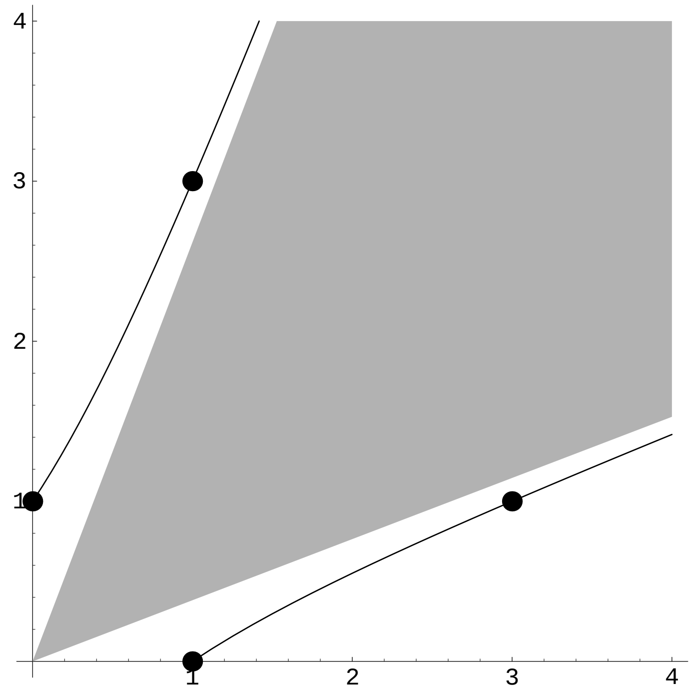

Figure 7: Schur roots of the Kronecker quiver.

Finally the real roots are obtained by the action of the Weyl group upon the fundamental roots. They correspond to the integer solutions of the equation $1 = n_1^2 + n_2^2 - 3n_1n_2$, which is just another way to say that they have $d = 0$. Explicitly, they are given by $(a_i\ a_{i+1})$ and $(a_{i+1}\ a_i)$ with

$$a_{i+2} = 3a_{i+1} - a_i\,, \qquad a_1 = 0 \qquad \text{and} \qquad a_2 = 1\,. \tag{A.18}$$

We should also mention that the dimension vectors for which the expected dimension formula predicts a negative dimension correspond to representations which are direct sums of (real) roots, so they are not indecomposable.

Weyl reflections appear in the discussion not just as symmetries of the Cartan matrix; in fact there are explicit "reflection operations" which take a representation and produce another representation with the Weyl reflected dimension vector.

Having found the set of positive roots, by the Kac theorem we have the set of dimension vectors for indecomposable representations. In general our next task will be to decide which of these correspond to Schur roots. The only characterization of Schur roots we are aware of is the following theorem due to Schofield, valid for quivers without relations [58]

$$\alpha \ \text{ is Schur} \iff \langle \beta, \alpha \rangle - \langle \alpha, \beta \rangle \ > \ 0 \qquad \text{for all } \beta \text{ subvector of } \alpha\,. \tag{A.19}$$

In this simple example we can bypass this theorem, since Kac [57] was able to prove that in this case all indecomposable roots are Schur (see also [59]). This is summarized in figure 7.

After determining the Schur roots of this quiver, we can study their θ-stability. Using Schofield's theorem (A.19), we learn that in this case $(n_1\ n_2)$ is Schur if and only if $n_1'/n_2' <$

n_1/n_2 for all its subvectors $(n'_1\ n'_2)$. Introduce now a vector $(\theta_1\ \theta_2)$. A Schur root will be θ-stable iff $n_1\theta_1 + n_2\theta_2 = 0$ and $n'_1\theta_1 + n'_2\theta_2 > 0$. Using the characterization of Schur roots just given, this amounts to saying that a Schur root of the Kronecker quiver is θ-stable iff $\theta_1 < 0$.

A.5 Quivers with relations

We consider a general set of relations, indexed by r, each expressed as a weighted sum over paths starting at ir and ending at tr.

Many of the basic definitions above generalize directly to this case. In particular, there is an Euler form

$$\langle \alpha, \beta \rangle = \sum_v \alpha_v \beta_v - \sum_a \alpha_{ia} \beta_{ta} + \sum_r \alpha_{ir} \beta_{tr} \tag{A.20}$$

which stands in the same relation to the expected dimension as (A.8):

$$d = 1 + \sum_a \dim(ia) \dim(ta) - \sum_v \dim(v)^2 - \sum_r \dim(ir) \dim(tr)\,. \tag{A.21}$$

For the $\mathbb{P}^2$ Beilinson quiver, these forms are precisely the ones (3.11) and (3.10) which appeared in the main text.

However, there do not seem to be results as general as the Kac theorem for the general quiver with relations. What results there are almost all assume that the quiver contains no closed loops. The specific results we use for the $\mathbb{P}^2$ Beilinson quiver, such as the fact that the expected dimension is realized, are generally taken from [11, 60] and other works on bundles on $\mathbb{P}^2$.

This should probably not be too surprising as the definition is general enough to encompass a wide variety of problems in algebraic geometry and supersymmetric gauge theory. Of course we might still hope that the same structure of roots, the Weyl group and so on will be useful in more general problems. A basic point to check in this regard is whether (in a given problem) the expected dimension is in fact realized as an actual dimension of moduli space. In general, as we discussed for the $\mathbb{Z}_3$ quiver in section 4, this need not be true, because of higher cohomology or equivalently relations between relations.

B. A few notations from algebraic geometry

Some standard sheaves: $\mathcal{O}_X$ is the sheaf of holomorphic functions on X. Ω is the holomorphic cotangent bundle. $\Omega^n \equiv \Lambda^n \Omega$ is its n'th exterior (antisymmetric) power. Ω^d is the canonical line bundle (on a d-dimensional manifold).

On $\mathbb{P}^n$, $\mathcal{O}(1)$ is the "hyperplane bundle," whose sections have a zero on a single $\mathbb{P}^{n-1} \subset \mathbb{P}^n$. (Thus they are linear combinations of the homogeneous coordinates). $\mathcal{O}(n)$ is the n-fold tensor product (sections are homogeneous functions of degree n). $E(n)$ for a general sheaf E is $E \otimes \mathcal{O}(n)$.

For a bundle E, $H^n(M, E^* \otimes F)$ can be thought of as global holomorphic n-forms taking values in $E^* \otimes F$. The groups $\mathrm{Ext}^n(E, F)$ can be thought of as the sheaf (and more generally homological algebra) generalization of this. It will generally satisfy the same "topological" properties, such as the index theorem for $\chi = \sum(-1)^n \dim \mathrm{Ext}^n$, usually called the Riemann-Roch theorem or Grothendieck-Riemann-Roch theorem.

Serre duality relates $H^n(M,E) \cong H^{d-n}(M, E^* \otimes \Omega^d)^*$.

References

[1] M.R. Douglas, *Topics in d-geometry*, *Class. and Quant. Grav.* **17** (2000) 1057 [hep-th/9910170].

[2] P. Kaste, W. Lerche, C.A. Lutken and J. Walcher, *D-branes on K3-fibrations*, *Nucl. Phys.* **B 582** (2000) 203 [hep-th/9912147].

[3] M. Naka and M. Nozaki, *Boundary states in Gepner models*, *JHEP* **05** (2000) 027 [hep-th/0001037].

[4] E. Scheidegger, *D-branes on some one- and two-parameter Calabi-Yau hypersurfaces*, *JHEP* **04** (2000) 003 [hep-th/9912188].

[5] I. Brunner and V. Schomerus, *D-branes at singular curves of Calabi-Yau compactifications*, *JHEP* **04** (2000) 020 [hep-th/0001132].

[6] B.R. Greene and C.I. Lazaroiu, *Collapsing D-branes in Calabi-Yau moduli space, I*, *Nucl. Phys.* **B 604** (2001) 181 [hep-th/0001025];
C.I. Lazaroiu, *Collapsing D-branes in one-parameter models and small/large radius duality*, *Nucl. Phys.* **B 605** (2001) 159 [hep-th/0002004].

[7] S. Govindarajan and T. Jayaraman, *On the Landau-Ginzburg description of boundary CFTS and special lagrangian submanifolds*, *JHEP* **07** (2000) 016 [hep-th/0003242].

[8] T.M. Chiang, A. Klemm, S.T. Yau and E. Zaslow, *Local mirror symmetry: calculations and interpretations*, *Adv. Theor. Math. Phys.* **3** (1999) 495 [hep-th/9903053].

[9] P.S. Aspinwall, B.R. Greene and D.R. Morrison, *Measuring small distances in $N = 2$ sigma models*, *Nucl. Phys.* **B 420** (1994) 184 [hep-th/9311042].

[10] D.-E. Diaconescu and J. Gomis, *Fractional branes and boundary states in orbifold theories*, *JHEP* **10** (2000) 001 [hep-th/9906242].

[11] J. Le Potier, *Lectures on vector bundles*, Cambridge University Press, Cambridge 1997

[12] A.A. Beilinson, *Coherent sheaves on $\mathbb{P}^n$ and problems of linear algebra*, *Funct. Anal. Appl.* **12** (1978) 214.

[13] M.R. Douglas, B.R. Greene and D.R. Morrison, *Orbifold resolution by D-branes*, *Nucl. Phys.* **B 506** (1997) 84 [hep-th/9704151].

[14] A.D. King, *Moduli of representations of finite dimensional algebras*, *Quart. J. Math. Oxford* (2), **45** (1994) 515.

[15] M.R. Douglas, B. Fiol and C. Romelsberger, *Stability and BPS branes*, hep-th/0002037.

[16] I. Brunner, M.R. Douglas, A.E. Lawrence and C. Romelsberger, *D-branes on the quintic*, *JHEP* **08** (2000) 015 [hep-th/9906200].

[17] M.R. Douglas, *D-branes and categories*, to appear.

[18] E. Calabi, *Métriques kählériennes et fibrés holomorhpes*, *Ann. Sci. Ec. Norm. Sup.* **12** (1979) 269.

[19] D.-E. Diaconescu and C. Romelsberger, *D-branes and bundles on elliptic fibrations*, *Nucl. Phys.* **B 574** (2000) 245 [hep-th/9910172].

[20] C.P. Bachas, P. Bain and M.B. Green, *Curvature terms in D-brane actions and their M-theory origin*, *JHEP* **05** (1999) 011 [hep-th/9903210].

[21] S.K. Donaldson and P.B. Kronheimer, *The geometry of four-manifolds*, Oxford University Press, Oxford 1990.

[22] J. Harvey and G. Moore, *On the algebras of BPS states*, *Commun. Math. Phys.* **197** (1998) 489.

[23] R. Friedman, *Algebraic surfaces and holomorphic vector bundles*, Springer Verlag, Berlin 1998.

[24] C. Okonek, M. Schneider and H. Spindler, *Vector bundles on complex projective spaces*, Birkhäuser, Boston 1980.

[25] K. Hulek, *On the classification of stable rank-r vector bunbles over the projective plane*, in proceedings of *Vector bundles and differential equations*, Nice 1979, pp. 113–144, *Progr. Math.* **7** Birkhäuser, Boston 1980.

[26] W. Barth and K. Hulek, *Monads and moduli of vector bundles*, *Manuscripta Math.* **25** (1978), no. 4, 323.

[27] E. Witten, *Phases of $N = 2$ theories in two dimensions*, *Nucl. Phys.* **B 403** (1993) 159 [hep-th/9301042].

[28] P. Griffiths and J. Harris, *Principles of algebraic geometry*, John Wiley & Sons Inc., 1994.

[29] M.R. Douglas and G.W. Moore, *D-branes, quivers and ale instantons*, hep-th/9603167.

[30] A.V. Sardo Infirri, *Partial resolutions of orbifold singularities via moduli spaces of hym-type bundles*, alg-geom/9610004.

[31] N. Seiberg and E. Witten, *Electric-magnetic duality, monopole condensation and confinement in $N = 2$ supersymmetric Yang-Mills theory*, *Nucl. Phys.* **B 426** (1994) 19 [hep-th/9407087], erratum *ibid.* **B430** (1994) 485.

[32] M. Marinõ, R. Minasian, G.W. Moore and A. Strominger, *Nonlinear instantons from supersymmetric p-branes*, *JHEP* **01** (2000) 005 [hep-th/9911206].

[33] S.I. Gel'fand and Y.I. Manin, *Methods of homological algebra*, Springer Verlag, Berlin 1996.

[34] M. Kontsevich, *Homological algebra of mirror symmetry*, alg-geom/9411018.

[35] R.P. Thomas, *Derived categories for the working mathematician*, math.AG/0001045.

[36] E.R. Sharpe, *D-branes, derived categories and Grothendieck groups*, *Nucl. Phys.* **B 561** (1999) 433 [hep-th/9902116].

[37] A. Tomasiello, *Projective resolutions of coherent sheaves and descent relations between branes*, *Adv. Theor. Math. Phys.* **4** (2002) 617 [hep-th/9908009].

[38] W. Lerche, *Introduction to Seiberg-Witten theory and its stringy origin*, *Nucl. Phys.* **55B** *(Proc. Suppl.)* (1997) 83 [*Fortschr. Phys.* **45** (1997) 293] [hep-th/9611190].

JHEP09(2005)057

[39] P. Seidel and R.P. Thomas *Braid group actions on derived categories of coherent sheaves*, math.AG/0001043 and to appear.

[40] Y. Ito and H. Nakajima, *McKay correspondence and Hilbert schemes in dimension three*, math.AG/9803120.

[41] R. Friedman, J. Morgan and E. Witten, *Vector bundles over elliptic fibrations*, alg-geom/9707029.

[42] A. Recknagel and V. Schomerus, *D-branes in Gepner models*, *Nucl. Phys.* **B 531** (1998) 185 [hep-th/9712186].

[43] I. Brunner and V. Schomerus, to appear.

[44] R.P. Horja, *Hypergeometric functions and mirror symmetry in toric varieties*, math.AG/9912109.

[45] A. Klemm, W. Lerche, P. Mayr, C. Vafa and N.P. Warner, *Self-dual strings and $N = 2$ supersymmetric field theory*, *Nucl. Phys.* **B 477** (1996) 746 [hep-th/9604034];
A. Sen and B. Zwiebach, *Stable non-BPS states in F-theory*, *JHEP* **03** (2000) 036 [hep-th/9907164];
T. Hauer and A. Iqbal, *Del pezzo surfaces and affine 7-brane backgrounds*, *JHEP* **01** (2000) 043 [hep-th/9910054];
K. Hori, A. Iqbal and C. Vafa, *D-branes and mirror symmetry*, hep-th/0005247.

[46] C. Vafa and E. Witten, *A strong coupling test of S duality*, *Nucl. Phys.* **B 431** (1994) 3 [hep-th/9408074].

[47] S. Katz, A. Klemm and C. Vafa, *M-theory, topological strings and spinning black holes*, *Adv. Theor. Math. Phys.* **3** (1999) 1445 [hep-th/9910181].

[48] D.-E. Diaconescu, M.R. Douglas and D.R. Morrison, to appear.

[49] C. Angelantonj, M. Bianchi, G. Pradisi, A. Sagnotti and Y.S. Stanev, *Chiral asymmetry in four-dimensional open-string vacua*, *Phys. Lett.* **B 385** (1996) 96 [hep-th/9606169].

[50] M. Cvetič, L.L. Everett, P. Langacker and J. Wang, *Blowing-up the four-dimensional Z_3 orientifold*, *JHEP* **04** (1999) 020 [hep-th/9903051].

[51] M.M. Kapranov, *On the derived categories of coherent sheaves on some homogeneous spaces*, *Invent. Math.* **92** (1988) 479.

[52] E. Witten, *Small instantons in string theory*, *Nucl. Phys.* **B 460** (1996) 541 [hep-th/9511030].

[53] D.J. Benson, *Representations and cohomology*, Cambridge University Press, Cambridge 1991.

[54] H. Kraft and C. Riedtmann, *Geometry of representations of quivers*, London Math. Soc. Lecture Note Ser., **116**, pp. 109-145, Cambridge University Press, Cambridge 1986.

[55] Y.-H. He, *Some remarks on the finitude of quiver theories*, hep-th/9911114.

[56] M.A. Luty and I.W. Taylor, *Varieties of vacua in classical supersymmetric gauge theories*, *Phys. Rev.* **D 53** (1996) 3399 [hep-th/9506098].

[57] V. Kac, *Infinite root systems, representations of graphs and invariant theory*, *Invent. Math.* **56** (1980) 57.

JHEP09(2005)057

[58] A. Schofield, *General representations of quivers*, *Proc. London Math. Soc.* (3) **65** (1992) no. 1, 46.

[59] A. Schofield, *Birational classification of moduli spaces of representations of quivers*, `math.AG/9911014`.

[60] A. Schofield, *Birational classification of moduli spaces of vector bundles over* $\mathbb{P}^2$, `math.AG/9912005`.